

HW#2. due 4/20/05. 1.1, 1.3, 1.5, 1.6, 1.8, 1.11.

Lecture 3

4/6/2005

Non-Markovian Example

$$\{\xi_n\}_{n=1}^{\infty} \text{ i.i.d.}$$

$$P(\xi_n = +1) = P(\xi_n = 0) = P(\xi_n = -1) = \frac{1}{3}$$

$$S_n = \sum_{i=1}^n \xi_i, \quad n = 0, 1, 2, \dots \quad (S_0 \equiv 0) \Rightarrow \text{So far, this is a Markov chain (random walk).}$$

$$\text{Define } Z_n = \max\{S_0, S_1, \dots, S_n\}, \quad n = 0, 1, 2, \dots$$

$$P(Z_{n+1} = j \mid Z_0 = i_0, \dots, Z_n = i_n) \stackrel{?}{=} P(Z_{n+1} = j \mid Z_n = i_n)$$

$\Rightarrow$ This is not Markov. When ^{proving the} checking a process is ^(not) Markov or ~~not~~, want to keep n small, i.e. when does the process become non-Markov?

Hitting times

Let $\{X_n\}_{n=0}^{\infty}$ be a Markov chain with discrete state space $\mathcal{S}$

Let $A \in \mathcal{S} = \mathcal{P}(\mathcal{S}) =$ all subsets of $\mathcal{S}$ (power set).

$$T_A = \inf\{n \geq 0 : X_n \in A\} = \begin{cases} (\text{if } \{n \geq 0 : X_n \in A\} \neq \emptyset) = \min\{n \geq 0 : X_n \in A\} \\ (\text{if } \{n : X_n \in A\} = \emptyset) = +\infty \end{cases}$$

(infimum of an empty set is infinity).

$$\tilde{T}_A = \inf\{n \geq 1 : X_n \in A\}$$

$$\text{e.g. if } X_0 = x, \quad \tilde{T}_x = \inf\{n \geq 1 : X_n \in A\}$$

= first return time to x .

(implicitly assuming $X_1 \neq x$)

Hitting times are stopping times.

$$\{T_A = n\} = \{\omega \in \Omega : T_A(\omega) = n\} \quad \text{depends only on } X_0, X_1, \dots, X_n$$

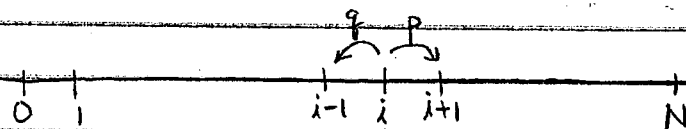
$$\{T_A = n\} = \{X_0 \notin A, X_1 \notin A, \dots, X_{n-1} \notin A, X_n \in A\}$$

Example

Random walk on $\{0, 1, \dots, N\} = S$

Simple random walk, $\{\xi_i\}_{i=1}^{\infty}$ i.i.d. $P(\xi_i = +1) = p$
 $P(\xi_i = -1) = q = 1-p$
 $0 < p < 1$

$$(*) \quad \begin{cases} X_n = x + \sum_{i=1}^n \xi_i, & x \in \{0, 1, \dots, N\} \\ \text{until hit one of end points} \end{cases}$$



$$(**) \quad \begin{cases} P_{ij} = P(X_{n+1} = j | X_n = i), & i \in \{1, \dots, N-1\} \\ = \begin{cases} q, & j = i-1 \\ p, & j = i+1 \\ 0, & \text{otherwise} \end{cases} \\ P_{0j} = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{else} \end{cases} \\ P_{Nj} = \begin{cases} 1 & \text{if } j = N \\ 0 & \text{else} \end{cases} \end{cases}$$

(*) and (**) are two ways to describe the Markov chain.

$$T_{0N} = \inf \{n \geq 0 : X_n = 0 \text{ or } N\}, \quad T_0 = \inf \{n \geq 0 : X_n = 0\}, \quad \dots$$

$$T_N = \inf \{n \geq 0 : X_n = N\}$$

$$T_{0N} = T_0 \wedge T_N$$

Questions:

(i) Is $T_{0N} < \infty$, with probability 1, or almost surely (a.s.), given $X_0 = i$, $i = 0, \dots, N$?

(ii) Is $E_i[T_{0N}] < \infty$ given $X_0 = i$, $i = 0, \dots, N$?

$$\Downarrow$$

$$E[T_{0N} | X_0 = i]$$

(iii) How to compute $P_i(T_0 < T_N)$ and $E_i[T_{0N}]$?

Answers:

(i), (ii) $\rightarrow$ Yes. (which will be shown using estimates)

$$(i) \quad P_i(\cdot) = P(\cdot | X_0 = i)$$

$$\begin{aligned} & P_i(T_{0N} \leq N-i), \quad (i = 1, \dots, N-1) \\ & \geq P_i(X_1 = i+1, X_2 = i+2, \dots, X_{N-i} = N) \\ & = \underbrace{P_{i,i+1} P_{i+1,i+2} \dots P_{N-i,N}}_{(N-i) \text{ terms}} = p^{N-i} \geq p^{N-1} \quad (i \in \{1, \dots, N\}) \quad (*) \end{aligned}$$

$$\left(\begin{array}{l} \text{If } i = 0, N: \\ P_i(T_{0N} \leq N-i) = \begin{cases} P_0(T_{0N} \leq N) = 1, & i = 0 \\ P_N(T_{0N} \leq 0) = 1, & i = N \end{cases} \\ = 1. \end{array} \right)$$

thus (*) true for $i \in \{0, 1, \dots, N\}$

$$\text{Let } p^{N-1} = 1 - \delta$$

$$\delta = 1 - p^{N-1} \in (0, 1)$$

$$P_i(T_{ON} > N) = 1 - P_i(T_{ON} \leq N) \leq 1 - P_i(T_{ON} \leq N-i) \leq \delta < 1 \quad \forall i \in \{0, \dots, N\}$$

(since $P_i(T_{ON} \leq N) \geq P_i(T_{ON} \leq N-i)$).

⇒ Can use Markov property to iterate and show geometric convergence.
(see below)

Claim (*) $P_i(T_{ON} > kN) \leq \delta^k$ for $i \in \{0, 1, \dots, N\}$, $k=1, 2, \dots$
(prove by induction)

True for $k=1$.

Assume (*) holds with $k-1$ in place of k for some $k-1 \geq 1$.

$$P_i(T_{ON} > kN) \quad i \in \{0, 1, \dots, N\}$$

$$= P_i(T_{ON} > kN \mid \underbrace{T_{ON} > N}_{\text{conditioning event}}) \cdot \underbrace{P_i(T_{ON} > N)}_{\leq \delta} \quad (\text{Bayes rule})$$

$$\left(\frac{\sum_j P_i(T_{ON} > kN, X_N=j, T_{ON} > N)}{P_i(T_{ON} > N, X_N=j)} \cdot \frac{P_i(X_N=j, T_{ON} > N)}{P_i(T_{ON} > N)} = P_i(T_{ON} > kN \mid T_{ON} > N) \right)$$

$$= \sum_j P_i(T_{ON} > kN \mid T_{ON} > N, X_N=j) \cdot P_i(X_N=j \mid T_{ON} > N) \cdot P_i(T_{ON} > N)$$

$$P_i(T_{ON} > kN \mid T_{ON} > N, X_N=j)$$

$\{T_{ON} = k\}$ depends only on $X_0, X_1, \dots, X_k$

$\{T_{ON} \leq k\}$ " " " " " "

$\{T_{ON} \leq N\}$ " " " $X_0, X_1, \dots, X_N$

(If we know an event happens or not, then we know if its complement happens or not).

$\{T_{ON} > N\}$ depends only on $X_0, X_1, \dots, X_N$
complement of $\{T_{ON} \leq N\}$

$$\begin{aligned}
 & P_i(T_{0N} > kN \mid T_{0N} > N, X_N = j) \\
 &= P_i(X_{N+1} \notin \{0, N\}, X_{N+2} \notin \{0, N\}, \dots, X_{kN} \notin \{0, N\} \mid X_N = j, \underbrace{T_{0N} > N}_{\text{depends on } X_0, X_1, \dots, X_N}) \\
 &= P_i(X_{N+1} \notin \{0, N\}, \dots, X_{kN} \notin \{0, N\} \mid X_N = j) \quad (\text{using Markov property}) \\
 &= P_j(X_1 \notin \{0, N\}, \dots, X_{(k-1)N} \notin \{0, N\}) \quad (\text{time homogeneity: shifting time}) \\
 &= P_j(T_{0N} > (k-1)N)
 \end{aligned}$$

$$\boxed{P_i(T_{0N} > kN \mid T_{0N} > N, X_N = j) = P_j(T_{0N} > (k-1)N)}$$

$$P_i(T_{0N} > kN), \quad i \in \{0, 1, \dots, N\}$$

$$= \sum_j \underbrace{P_j(T_{0N} > (k-1)N)}_{\text{use induction assumption}} P_i(X_N = j \mid T_{0N} > N) \cdot \underbrace{P_i(T_{0N} > N)}_{\leq \delta}$$

$$\leq \sum_j \delta^{k-1} P_i(X_N = j \mid T_{0N} > N) \cdot \delta$$

$$\leq \delta^k$$

$$\begin{aligned}
 & P_i(T_{0N} > kN) \leq \delta^k \longrightarrow 0 \quad \text{as } k \longrightarrow \infty \\
 & \Rightarrow T_{0N} < \infty \quad \text{a.s.} \quad (T_{0N} \text{ is finite})
 \end{aligned}$$

$$\begin{aligned}
 (ii) \quad E_i[T_{0N}] &= \sum_{l=0}^{\infty} P_i(T_{0N} > l) \\
 &\leq \sum_{k=1}^{\infty} \sum_{l=0}^{N-1} \underbrace{P_i(T_{0N} > (k-1)N)}_{\leq \delta^{k-1}} \leq \sum_{k=1}^{\infty} N \delta^{k-1} < \infty
 \end{aligned}$$

non-negative integers.

(Break into blocks of size N)

(undergrad probability trick: $X \geq 0$, integer valued, $E[X] = \sum_{i=0}^{\infty} P(X > i)$)