

LECTURE 2

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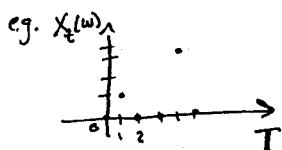
- Lecture 1 notes are on web page
 - HW #1 due next Wed (04/06) in class
- Today - ways to think about stochastic processes
- Markov chains, ch 0.1.2

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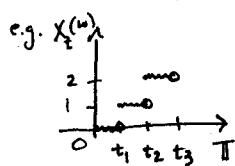
Stochastic process: $\{X_t, t \in T\}$ defined on some probability space $(\Omega, \mathcal{F}, P)$

Views of stochastic processes:

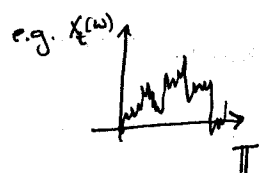
1. For each $t \in T$, $X_t: \Omega \rightarrow \mathcal{S}$ "observing outcome at each time t "
2. For each $\omega \in \Omega$, $t \mapsto X_t(\omega)$ "evolution of outcomes over time"
(sample path) $T \rightarrow \mathcal{S}$



Discrete time
Discrete state
(count state)



cont time: $T = [0, \infty)$
Discrete state: $\mathcal{S} = \mathbb{N}$
(eg. Poisson process)



cont time
cont state
(eg. Brownian Motion)

3. think of as fun of $t \in T$, $X: T \times \Omega \rightarrow \mathcal{S}$
 $X(t, \omega) = X_t(\omega)$

(useful for certain integrations)

Distribution of a Stochastic process (law of stochastic process)

Single random variable $Y: \Omega \rightarrow \mathcal{S}$

Specify $P(Y \in A)$ for each $A \in \mathcal{G} \leftarrow \sigma\text{-alg generated by } \mathcal{S}$ on $\mathcal{S}$

This gives a distribution

eg. if $\mathcal{S} = \mathbb{R}$, cum. dist. fun

Specify $F_Y(x) = P(Y \leq x) = P(Y \in (-\infty, x])$ for each $x \in \mathbb{R}$

i.e. look at generating sets for $\sigma\text{-alg}$ on $\mathbb{R}$.

eg. if $\mathcal{S} = \mathbb{N}$

$P(Y = n)$, for each $n \in \mathbb{N}$

For a stochastic process $\{X_t, t \in T\}$ the distribution is determined by specifying all of the finite dimensional distributions:

take $\{t_1, \dots, t_n\} \subset T$ specify $P(X_{t_1} \in A_1, \dots, X_{t_n} \in A_n)$

for each $A_1, \dots, A_n \in \mathcal{G}$

eg. $\mathcal{S} = \mathbb{R}$

take $A_i = (-\infty, x_i]$, $i = 1, \dots, n$

specify $P(X_{t_1} \leq x_1, \dots, X_{t_n} \leq x_n) = F_{t_1, \dots, t_n}(x_1, \dots, x_n)$

↑

Multi-dim distribution fun.

Distribution of stochastic process doesn't tell you anything about sample path behavior.

Markov Chains

State space is discrete.

distribution at future times (assumes there is a natural ordering of index set) depends on past only thru the current state.

i.e. $P(X_t \in A | X_s = s)$ for $s < t$, $A \in \mathcal{S}$

$$P(X_t \in A | X_u: u \leq s) = P(X_t \in A | X_s = s)$$

DISCRETE TIME MARKOV CHAINS

DEF: A discrete time Markov chain is a stochastic process $\{X_n, n=0,1,2,\dots\}$ with discrete state space $\mathcal{S}$ s.t.

$$(*) P(X_n = i_n | X_0 = i_0, \dots, X_{n-1} = i_{n-1}) = P(X_n = i_n | X_{n-1} = i_{n-1})$$

for any $i_0, i_1, \dots, i_{n-1}, i_n \in \mathcal{S}$, $n=1,2,\dots$

Recall: $P(A|B) = \frac{P(A \cap B)}{P(B)}$ ok if $P(B) \neq 0$.

Note:

1. If the conditioning event in LHS of (*) has probability zero and the one in RHS does not then define LHS = RHS
2. If both conditioning events have prob. zero define both sides of (*) to be zero.

One-step transition probability

$$P_{i,j}^{n-1,n} := P(X_n = j | X_{n-1} = i) \text{ for } i,j \in \mathcal{S}, n=1,2,\dots$$

Can recover distribution of $\{X_n, n=0,1,2,\dots\}$ from one-step transition probs plus initial distribution ϕ where $\phi(i) = P(X_0 = i)$, $i \in \mathcal{S}$.

For any n and $i_0, \dots, i_n \in \mathcal{S}$,

$P(X_0 = i_0, \dots, X_n = i_n)$ $\leftarrow$ there specify distribution $\nwarrow$ unconditional

$$= P(X_n = i_n | X_{n-1} = i_{n-1}, \dots, X_0 = i_0) \cdot P(X_{n-1} = i_{n-1}, \dots, X_0 = i_0)$$

$$= P(X_n = i_n | X_{n-1} = i_{n-1}, \dots, X_0 = i_0) P(X_{n-1} = i_{n-1} | X_{n-2} = i_{n-2}, \dots, X_0 = i_0)$$

$$\dots P(X_1 = i_1 | X_0 = i_0) P(X_0 = i_0)$$

$$= \phi(i_0) \prod_{k=1}^n P_{i_{k-1}, i_k}$$

$$= P(X_n = i_n | X_{n-1} = i_{n-1}) P(X_{n-1} = i_{n-1} | X_{n-2} = i_{n-2}) \dots P(X_1 = i_1 | X_0 = i_0) \phi(i_0)$$

$P(X_0 = i_0)$

- by Markov prop.

$$= P_{i_0, i_1}^{n-1, n} P_{i_1, i_2}^{n-2, n-1} \dots P_{i_{n-1}, i_n}^{0,1} \phi(i_0)$$

Time Homogeneous

$P_{i,j}^{n-1,n}$ does not depend on n

$$P_{i,j} \quad i,j \in \mathcal{S}$$

these determine a matrix (square "matrix") since $\mathcal{S}$ is discrete

$P = (P_{i,j})_{i,j \in \mathcal{S}}$ is a square matrix (could have infinitely many rows + cols)

Note: 1. $P_{i,j} \geq 0 \quad \forall i,j \quad \& \quad P_{i,j} \leq 1$

2. Each row sum is 1, since

$$\sum_j P_{i,j} = \sum_j P(X_1 = j | X_0 = i) = P(\cup \{X_1 = j\} | X_0 = i) = 1$$

since X_1 has to equal something in $\mathcal{S}$

Can compute n-step transition probability

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n step prob: $P(X_n = j | X_0 = i)$ $i, j \in S$

$= (P^n)_{ij}$ (prove this using induction)

e.g. check this holds for $n=2$

$$\begin{aligned}
 P(X_2 = j | X_0 = i) &= P(X_2 = j, \bigcup_k \{X_1 = k\} | X_0 = i) \\
 &= P(X_2 = j, X_1 \in S | X_0 = i) \\
 &= \sum_k P(X_2 = j, X_1 = k | X_0 = i) \\
 &= \sum_k P(X_2 = j | X_1 = k, X_0 = i) P(X_1 = k | X_0 = i) \\
 \underline{MP} \rightarrow &= \sum_k P(X_2 = j | X_1 = k) P(X_1 = k | X_0 = i) \\
 &= \sum_k P_{kj} P_{ik} \\
 &= \sum_k P_{ik} P_{kj} = (P^2)_{ij} \quad \checkmark
 \end{aligned}$$

Examples:

1. Random walk on a graph

Finite Graph (V, E)

finite set of vertices

edges (each edge connects two vertices) (finitely many) (no duplicates)



X_n = position at time n

$d(i)$ = degree of vertex i

= # edges directly emanating from i

$P(X_n = j | X_{n-1} = i)$ $j \sim i$ means j is connected directly to i

$$P(X_n = j | X_{n-1} = i) = \begin{cases} \frac{1}{d(i)} & j \sim i \\ 0 & j \not\sim i \end{cases}$$

this is Markov.

2. Branching Process

X_n = # of individuals in generation n

with $X_0 = 1$

For generation n , specify $\{\xi_i^{(n)}, i=1,2,\dots\}$ i.i.d. nonneg integer valued random variables

Independence of $\{\xi_i^{(n)}\}_{i=1}^\infty$ as n varies.

$$X_n = \sum_{i=1}^{X_{n-1}} \xi_i^{(n)}$$

Interpret $\xi_i^{(n)}$ as # individuals created as offspring by individual i from generation $n-1$

$$P(X_n = j | X_0 = i_0, \dots, X_{n-1} = i_{n-1})$$

$\xi_i^{(m)}, m=0,1,\dots,n-3$

$$= P(X_n = j | X_{n-1} = i_{n-1})$$

as X_n determined by $X_{n-1} + \xi_i^{(n-1)}$ & knowing $X_0, \dots, X_{n-2}$ does not provide further information

Questions for Markov chains

- Finite time horizon, transient questions -
 - first passage times
 - hitting probabilities
- Long Run behavior
 - transience - running "off to infinity" (if S is countable)
 - recurrence: visiting states - how often?