

Lost Time:

B is a standard 1 dim Brownian Motion

$$\mathcal{F}_t = \{ B_s : 0 \leq s \leq t \}$$

i) $\{ B_t, \mathcal{F}_t, t \geq 0 \}$ is a martingale

ii) $\{ B_t^2 - t, \mathcal{F}_t, t \geq 0 \}$ is a martingale

Proof: a) $B_t^2 - t \in \mathcal{F}_t \quad \forall t \geq 0$

$$b) E[|B_t^2 - t|] \leq E[B_t^2] + t = t + t = 2t < \infty \quad \forall t \geq 0$$

$$c) E[B_t^2 - t | \mathcal{F}_s] = E[(B_t - B_s)^2 + 2B_s(B_t - B_s) + B_s^2 - t | \mathcal{F}_s]$$

$$= E[(B_t - B_s)^2] + 2B_s E[B_t - B_s | \mathcal{F}_s] + B_s^2 - t$$

$\uparrow$
 $B_t - B_s \perp \mathcal{F}_s$

$\uparrow$
 B_s is $\mathcal{F}_s$ measurable

$$= t - s + 2B_s \underset{=0}{E[B_t - B_s]} + B_s^2 - t$$

$$= B_s^2 - s$$

These are 2 important Martingale properties of Brownian Motion

* Brownian Motion is often a noise source in a stochastic DE

Example: $X(t)$ = state of the system at time $t \quad t \geq 0$

$$- \frac{dX(t)}{dt} = f(X(t)) + g(X(t)) W(t) \quad W(t) \text{ is called white noise}$$

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ODE + Stochastic Term

We can turn this into an integral equation

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$$X(t) = X(0) + \int_0^t f(X(s)) ds + \int_0^t g(X(s)) \underbrace{W(s) ds}_{B(s)}$$

Itô's lemma

$$dX(t) = f(X(t)) dt + g(X(t)) d(B(t))$$

↑ driving noise

Property of BM Scaling

for $c > 0$:

$\{ \frac{B_{ct}}{\sqrt{c}}, t \geq 0 \}$ is also a standard 1-dim Brownian Motion

// distribution

$\{ B_t, t \geq 0 \}$

Check the properties:

i) $\frac{B_{c \cdot 0}}{\sqrt{c}} = B_0$ as

ii) $\frac{B_{ct_1}}{\sqrt{c}} - \frac{B_{ct_0}}{\sqrt{c}}, \frac{B_{ct_2} - B_{ct_1}}{\sqrt{c}}, \dots, \frac{B_{ct_n} - B_{ct_{n-1}}}{\sqrt{c}}$

for $0 \leq t_0 < t_1 < t_2 < \dots < t_n < \infty$



Are independent since time intervals are disjoint

iii) for $0 \leq s < t < \infty$

$\frac{B_{ct} - B_{cs}}{\sqrt{c}} \sim N(0, t-s)$ Since $B_{ct} - B_{cs} \sim N(0, c(t-s))$

iv) Paths are still continuous $t \longrightarrow \frac{B_{ct}}{\sqrt{c}}$

Brownian Motion is a Gaussian Process

③

Def. $\{X(t), t \geq 0\}$ real valued process is a Gaussian process if for any $0 \leq t_1 < t_2 < \dots < t_n < \infty$ $(X(t_1), \dots, X(t_n))$ is a multivariate Gaussian

i.e. $\exists$ an $n \times n$ matrix A , a constant $\vec{b}$ and a vector $Y = (Y_1, \dots, Y_n)$ where $Y_i \sim N(0, 1)$ $i = 1, \dots, n$ and Y_i are independent st $\tilde{X} \stackrel{\text{dist}}{=} AY + \vec{b}$

Characterize distribution of X by

i) $m(t) = E(X(t))$

ii) $R(s, t) = E(X(s)X(t)) - E(X(s))E(X(t))$ $0 \leq s, t < \infty$

Brownian Motion is a Gaussian Process:

$m(t) = 0$, $R(s, t) = s \wedge t = \min(s, t)$

For any $0 \leq t_1 < \dots < t_n < \infty$

$$(B_{t_1}, \dots, B_{t_n}) = (B_{t_1}, (B_{t_2} - B_{t_1}) + B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}} + B_{t_{n-1}} - B_{t_{n-2}} + \dots + B_{t_2} - B_{t_1} + B_{t_1})$$

$= A \begin{pmatrix} B_{t_1} / \sqrt{t_1} \\ \frac{B_{t_2} - B_{t_1}}{\sqrt{t_2 - t_1}} \\ \vdots \\ \frac{B_{t_n} - B_{t_{n-1}}}{\sqrt{t_n - t_{n-1}}} \end{pmatrix}$ ← We can normalize these to be $N(0, 1)$

Where $A = \begin{pmatrix} \sqrt{t_1} & 0 & \dots \\ \sqrt{t_1} & \sqrt{t_2 - t_1} & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}$

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$m(t) = 0$ is clear

$$R(s, t) = E(B(s)B(t)) - \underbrace{E(B(s))}_{0} \underbrace{E(B(t))}_{0} \quad 0 \leq s < t < \infty$$

$$= E(B(s)(B(t) - B(s)) + B(s)^2)$$

$$= E(B(s)) \underbrace{E(B(t) - B(s))}_{0} + E(B(s)^2)$$

$$= s$$

Note. It is easy to characterize distribution - you only need 2 functions

* Brownian Motion is a Markov process

Markov Property (Simple):

$$\mathcal{F}_s = \sigma\{B_u : 0 \leq u \leq s\} \quad s \geq 0$$

fix $s \geq 0$, $t > s$, A a Borel set in $\mathbb{R}$

$$P(B_t \in A \mid \mathcal{F}_s) \stackrel{?}{=} P(B_t \in A \mid B_s)$$

let $A - x = \{y - x : y \in A\}$

$$P(B_t - B_s \in A - B_s \mid \mathcal{F}_s) = P(B_t - B_s \in A - B_s \mid B_s)$$

$$= P(B_t \in A \mid B_s)$$

Strong Form of Strong Markov Property

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We need a notion of Stopping Time in continuous time

Def. A Stopping Time relative to $\{\mathcal{F}_t, t \geq 0\}$ is a function $\tau: \Omega \longrightarrow [0, \infty]$ st $\{\tau \leq t\} = \{\omega \in \Omega: \tau(\omega) \leq t\} \in \mathcal{F}_t \quad \forall t \geq 0$

The idea of the Strong Markov Property:

You want to do simple Markov Property st we replace s by τ

Given a stopping time τ let $\mathcal{F}_\tau = \{A \in \mathcal{F}: \{\tau \leq t\} \cap A \in \mathcal{F}_t \quad \forall t \geq 0\}$

Note: $\mathcal{F}_\tau \neq \mathcal{F}_{\tau(\omega)}$

Strong Markov Property for BM

For any stopping time $\tau, t > 0, A \in \mathcal{B}(\mathbb{R})$ - Borel sets in $\mathbb{R}$

$$P(B_{t+\tau} \in A \mid \mathcal{F}_\tau) = P(B_{t+\tau} \in A \mid B_\tau) \quad \text{on } \{\tau < \infty\}$$

This is on $\{\tau < \infty\}$ for otherwise $B_{t+\tau}$ makes no sense

Standard trick for approximating τ



$$\text{for each } n \in \mathbb{N} \quad \text{let } \tau^n = \frac{\lfloor 2^n \tau + 1 \rfloor}{2^n}$$

(check stopping time property)

$\tau^n \downarrow \tau \leftarrow$ strictly decreases

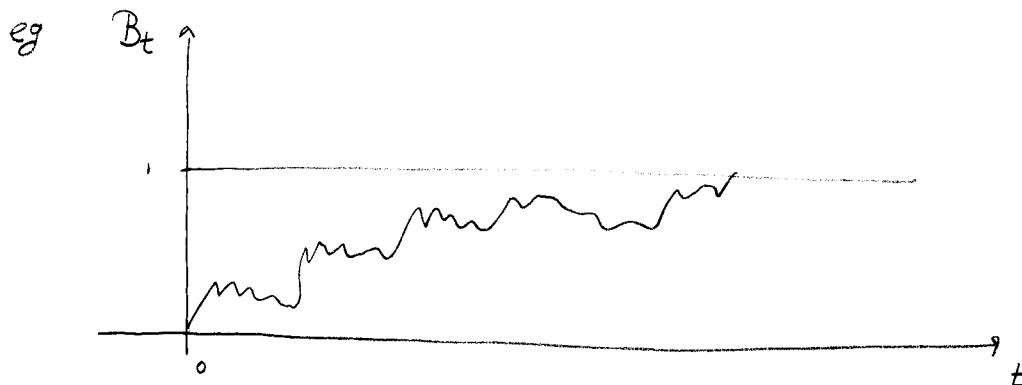
Now we can prove strong markov property from simple MP

Stopping Time for BM:

(6)

If A is a closed set in $\mathbb{R}$

then $\tau_A = \inf \{ t \geq 0 : B_t \in A \}$ = first entrance time to A
is a stopping time relative to $\{\mathcal{F}_t\}$



$$\tau_{[1, \infty)} = \inf \{ t \geq 0 : B_t \geq 1 \}$$

this is a stopping time

Brownian Motion Starting from $x \in \mathbb{R}$ is a stochastic process equivalent in distribution to $\{B_t + x, t \geq 0\}$

For $d \geq 1$, d -dimensional Brownian Motion starting from $x \in \mathbb{R}^d$ is equivalent to

$x + (B^1, \dots, B^d)$ where B^i are independent standard

BM's

Multidimensional BM inherits a lot of properties of 1-dim BM

Super - Strong Markov Property

⑦

Suppose τ is a stopping time with $\tau < \infty$ everywhere

$\{B_{t+\tau} - B_\tau, t \geq 0\}$ is independent of $\mathcal{F}_\tau$ and is a standard 1-dim BM

$$(B_{t+\tau})(\omega) = B_{t+\tau}(\omega_1(\omega))$$

$$(B_\tau(\omega_1)(\omega) = (B_\tau)(\omega)$$

Brownian Motion & Harmonic Analysis (PDE)



D bounded open set in $\mathbb{R}^d$

f is continuous on ∂D

$$\Delta u = 0 \text{ in } D$$

u smooth in D

$$u \in C^0(\bar{D} = D \cup \partial D)$$

$$\begin{cases} \Delta u = \sum_{i=1}^d \frac{\partial^2 u}{\partial x_i^2} = 0 \\ u = f \text{ on } \partial D \end{cases} (*)$$

How do you probabilistically solve the problem

$B = d$ -dim BM starting at x

let $\tau = \inf \{t \geq 0 : B_t \notin D\}$ is a stopping time

$f(B_\tau)$ - value of f where BM stops

$$\text{let } u(x) = E_x[f(B_\tau)]$$

If ∂D is regular u is the unique solution of (*)