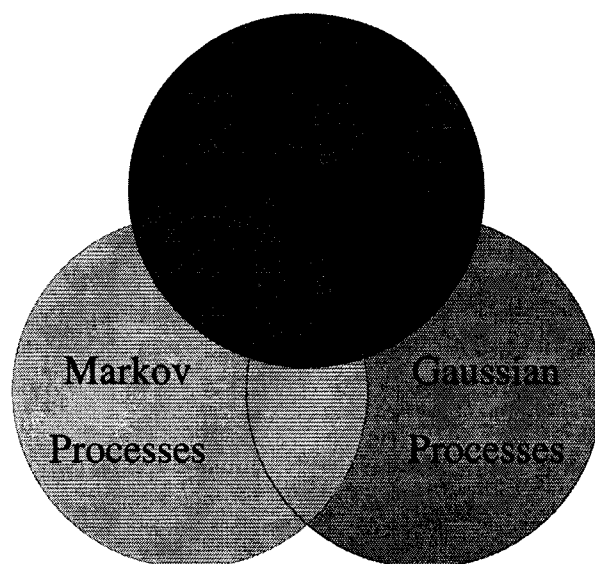


Math 285A Lecture Notes

BROWNIAN MOTION:

Continuous time, continuous state – e.g. $\mathbb{R}^d, d \geq 1. \{B_t, t \in R_+\}, R_+ = [0, \infty)$.



Define:

A continuous time stochastic process $\{B_t, t \geq 0\}$ is a one-dimensional standard Brownian motion if:

(I) $B_0 = 0$, a.s. (with probability 1)

(II) For any $0 \leq t_0 < t_1 < t_2 < \dots < t_n < \infty$, $(B_{t_n} - B_{t_{n-1}}, B_{t_{n-1}} - B_{t_{n-2}}, \dots, B_{t_1} - B_{t_0})$ are independent, in other words, independent increments.

(III) For any $0 \leq s < t < \infty$, $B_t - B_s$ is a $N(0, t-s)$ random variable.

(IV) as $t \rightarrow B_t$ is continuous. (Wiener)

Brownian motion has stationary independent increments.

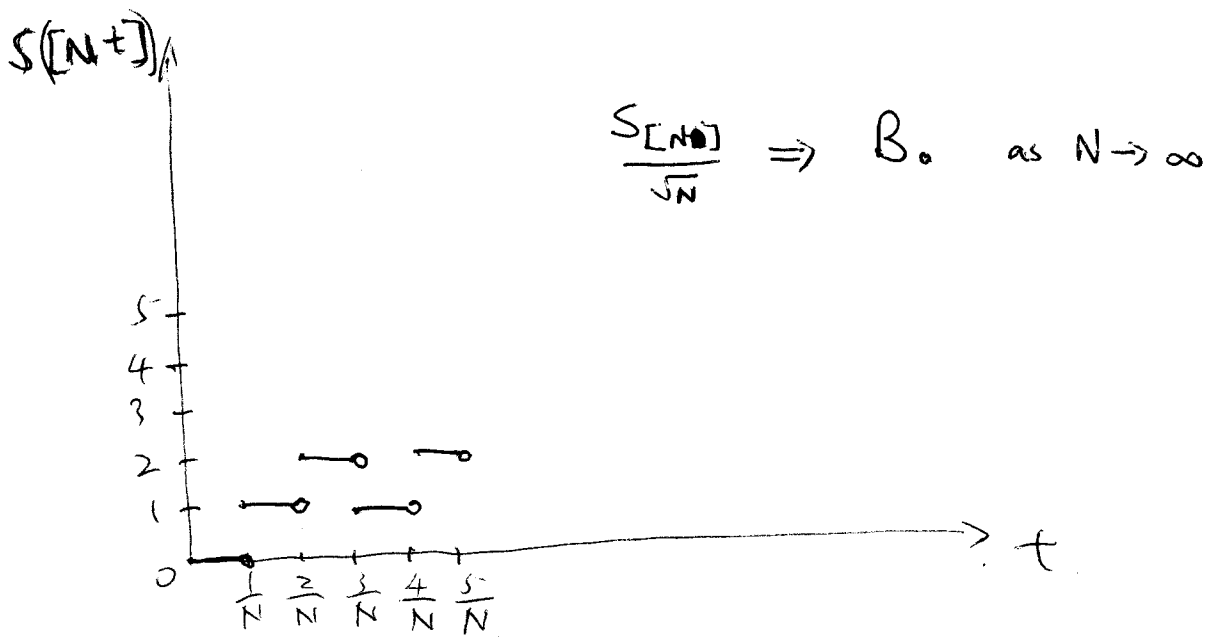
Limit of rescaled random walks.

$\{\xi_i\}_{i=1}^n$, i.i.d, mean 0, $\text{Var}(\xi_i) = E[\xi_i^2] = 1$. $P(\xi_i = +1) = P(\xi_i = -1) = \frac{1}{2}$.

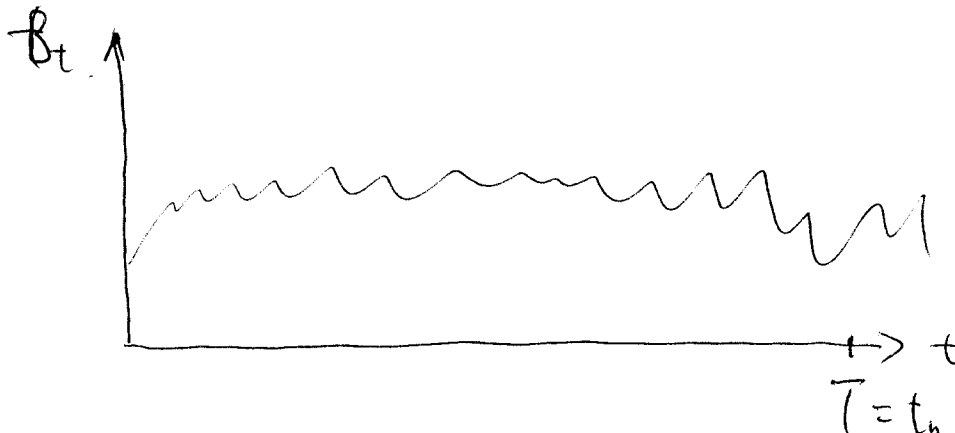
$$S_n = \sum_{i=1}^n \xi_i, n = 0, 1, 2, \dots$$

CLT

$$\frac{S_n}{\sqrt{n}} \Rightarrow X \sim N(0, 1) \leftarrow \text{normally distributed (mean 0, variance 1)}$$



Paths of Brownian motion are not of bounded variation



$$t_0 = 0 < t_1 < t_2 < \dots < t_n = T$$

$$V(t) = \sup_{\pi_n} \sum_{i=0}^{n-1} |B_{t_{i+1}} - B_{t_i}| = \text{variation of } B$$

$$Q^n(B) = \sum_{i=0}^{n-1} |B_{t_{i+1}} - B_{t_i}|^2, \text{ quadratic variation, } \sup_{\pi_n} Q^n(B) < \infty$$

Brownian motion paths are a.s. of finite quadratic variation, $t_i^n = \frac{i \cdot T}{n}, i = 0, 1, 2, \dots, n$

$$Q^n(B) = \sum_{i=0}^{n-1} |B_{t_{i+1}^n} - B_{t_i^n}|^2$$

$$\begin{aligned} E[(Q^n(B) - T)^2] &= E[(Q^n(B))^2 - 2TQ^n(B) + T^2] \\ &= E[(\sum_{i=0}^{n-1} (B_{t_{i+1}^n} - B_{t_i^n})^2 - 2T \sum_{i=0}^{n-1} (B_{t_{i+1}^n} - B_{t_i^n})^2 + T^2] \\ &= \sum_{i=0}^{n-1} E[(B_{t_{i+1}^n} - B_{t_i^n})^4] + \sum_{i=0}^{n-1} \sum_{j \neq i} E[(B_{t_{j+1}^n} - B_{t_j^n})^2 (B_{t_{i+1}^n} - B_{t_i^n})^2] - 2T \sum_{i=0}^{n-1} E[(B_{t_{i+1}^n} - B_{t_i^n})^2] + T^2 \\ &= \sum_{i=0}^{n-1} 3 \left(\frac{T}{n} \right)^2 + \sum_{i=0}^{n-1} \sum_{j \neq i} E[(B_{t_{j+1}^n} - B_{t_j^n})^2 (B_{t_{i+1}^n} - B_{t_i^n})^2] - 2TT + T^2 \\ &= 3n \left(\frac{T^2}{n^2} \right) + n(n-1) \frac{T}{n} \frac{T}{n} - T^2 = \frac{2T^2}{n}, \rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned}$$

$$Q^n(B) \rightarrow T \text{ in } L^2 \text{ as } n \rightarrow \infty$$

$$\exists \{n_k\} \text{ such that } Q^{n_k}(B) \rightarrow T \text{ a.s. as } n \rightarrow \infty$$

It follows that B does not have bounded variation paths a.s.

$$Q^{n_k}(B) = \sum_{i=0}^{n_k-1} |B_{t_{i+1}^{n_k}} - B_{t_i^{n_k}}|^2 \leq \underbrace{(\max_{i=0}^{n_k-1} |B_{t_{i+1}^{n_k}} - B_{t_i^{n_k}}|)}_{\rightarrow 0 \text{ as } n \rightarrow \infty} \underbrace{\sum_{i=0}^{n_k-1} |B_{t_{i+1}^{n_k}} - B_{t_i^{n_k}}|}_{\text{if converge to sth finite}}$$

then $Q^{n_k}(B) \rightarrow 0$ a.s. as $n \rightarrow \infty$ (can only happen on a set of probability 0).

a.s. the paths of Brownian motion are not differentiable anywhere.

Further Evidence for this:

$$\frac{B_{t+h} - B_t}{h} \stackrel{\text{distributional}}{=} \frac{B_h}{h} = \frac{\sqrt{h}B_1}{h} = \frac{B_1}{\sqrt{h}}, \text{ doesn't converge to anything as } h \rightarrow 0$$

MARTINGALE

Definition

A collection $\{F_t, t \geq 0\}$ of sub- σ -algebras of $\mathcal{F}$.

F is a filtration if $F_s \subseteq F_t$ for $0 \leq s \leq t < \infty$

For Brownian motion, take

$F_t = \sigma(B_s : 0 \leq s \leq t) \leftarrow$ natural filtration

= smallest σ -algebra containing all sets of the form $B_s^{-1}((a, b])$, $-\infty < a < b < +\infty$, $0 \leq s \leq t$.

DEFN

~~Defn~~ A continuous time real valued stochastic process $M = \{M_t, t \geq 0\}$ is a

martingale (with respect to filtration $\{F_t, t \geq 0\}$), if

(I) $M_t \in F_t$ (i.e. M_t is an F_t measurable random variable for each $t \geq 0$)

(II) $E[|M_t|] < \infty, \forall t \geq 0$

(III) $E[M_t | F_s] = M_s$ for $0 \leq s < t < \infty$

Then, Brownian motion is a martingale with respect to its natural filtration

Proof (III):

Fix $0 \leq s < t < \infty$

$$E[B_t | F_s] = E[B_t - B_s + B_s | F_s] = \underbrace{E[B_t - B_s | F_s]}_{(B_t - B_s) \perp F_s} + B_s = E[B_t - B_s] + B_s = 0 + B_s$$