

1 Optional Stopping (Sampling) Theorem

We assume a filtration $\{\mathcal{F}_n, n = 0, 1, 2, \dots\}$ where $\{\mathcal{F}_n\}$ is an increasing family of σ -algebras.

Def. 1.1. *Stopping time* (relative to filtration $\{\mathcal{F}_n\}$) is a random variable

$$T : \Omega \rightarrow \mathbb{N} \cup \{\infty\}$$

that satisfies the non-anticipating property, i.e.,

$$\{\omega \in \Omega : T = n\} \in \mathcal{F}_n \text{ for each } n \in \mathbb{N}.$$

As an example consider the asymmetric random walk from previous lecture

$$X_n = x + \sum_{i=1}^n \xi_i$$

where $\{\xi_i\}_{i=1}^\infty$ are independent and identically distributed (i.i.d.) random variables taking value ± 1 with $P\{\xi_i = +1\} = p$ and $P\{\xi_i = -1\} = q$ where $0 < p < 1$, $q = 1 - p$ and $p \neq q$, x is an integer.

Define $\mathcal{M}_n = X_n - n\mu$ where $\mu = \mathbf{E}[\xi_i]$ and filtration

$$\begin{aligned} \mathcal{F}_n &= \sigma\{\xi_1, \xi_2, \dots, \xi_n\} \\ &= \sigma\{X_0, X_1, \dots, X_n\}. \end{aligned}$$

Fix $0 < x < b < \infty$ where 0 and b are integers.

Define the stopping time T as

$$T = \inf\{n \geq 0 : X_n = 0 \text{ or } b\}.$$

We would like to define T as the “min” but we need to use “inf” to take care of the special case: $\inf \emptyset = +\infty$.

Fix $n \in \mathbb{N}$. Then,

$$\begin{aligned} \{\omega \in \Omega : T(\omega) = n\} &= \{\omega \in \Omega : X_0(\omega) \neq 0 \text{ or } b, X_1(\omega) \neq 0 \text{ or } b, \dots, \\ &\quad X_{n-1}(\omega) \neq 0 \text{ or } b, X_n(\omega) = 0 \text{ or } b\} \in \mathcal{F}_n. \end{aligned}$$

As an example of a random variable that is not a “stopping time” consider

$$S = \sup\{n \geq 0 : X_n = 0\}.$$

S is the last time $X_n = 0$. To know S , we need to know all the future because

$$\begin{aligned} \{S = n\} &= \{\omega \in \Omega : S(\omega) = n\} \\ &= \{X_n = 0, X_{n+1} \neq 0, X_{n+2} \neq 0, \dots\}. \end{aligned}$$

Can prove that S is not a stopping time.

Thm. 1.1 (Simple Optional Stopping Theorem). Suppose $\{\mathcal{M}_n, \mathcal{F}_n, n = 0, 1, 2, \dots\}$ is a discrete-time martingale and T is a stopping time with $T \leq K$ for some constant $K < \infty$. Then,

$$\mathbf{E}[\mathcal{M}_T] = \mathbf{E}[\mathcal{M}_0].$$

As an example consider the asymmetric random walk starting at x . We want to find the probability of getting to b before 0 and the expected value of T , $\mathbf{E}[T]$.

Then, $T = \inf\{n \geq 0 : X_n = 0 \text{ or } X_n = b\}$. Fix a large positive integer N . Then, the random variable $T \wedge N$ is a stopping time. Here $T \wedge N$ is the $\min\{T, N\}$.

Exercise: Verify that the minimum of two stopping times is a stopping time.

From previous lecture, $\{\mathcal{M}_n = X_n - n\mu, \mathcal{F}_n, n = 0, 1, 2, \dots\}$ is a martingale. By the stopping theorem,

$$\mathbf{E}[\mathcal{M}_{T \wedge N}] = \mathbf{E}[\mathcal{M}_0] = x$$

or

$$\mathbf{E}[X_{T \wedge N} - (T \wedge N)\mu] = x$$

or

$$\mu \mathbf{E}[T \wedge N] = \mathbf{E}[X_{T \wedge N}] - x. \quad (1)$$

But $X_{T \wedge N} \in [0, b]$ since $0 \leq X_{T \wedge N} \leq b$. We use this to show that T is finite, almost surely. Taking absolute value on both sides of (1)

$$|\mu \mathbf{E}[T \wedge N]| = |\mathbf{E}[X_{T \wedge N}] - x|$$

and using the triangle inequality

$$|\mu| \mathbf{E}[T \wedge N] \leq b + x$$

or

$$\mathbf{E}[T \wedge N] \leq \frac{b + x}{|\mu|}.$$

As $N \rightarrow \infty$, $T \wedge N \rightarrow T$, almost surely. By the monotone convergence theorem, letting $N \rightarrow \infty$ we obtain

$$\mathbf{E}[T] \leq \frac{b + x}{\mu} < \infty.$$

Hence, $T < \infty$, almost surely.

Note that, since $X_{T \wedge N}$ is bounded, we can let $N \rightarrow \infty$ in (1), noting that $X_{T \wedge N} \rightarrow X_T$ a.s., to conclude by the bounded convergence theorem that

$$\mu \mathbf{E}[T] = \mathbf{E}[X_T] - x.$$

Next, we use the other martingale defined by

$$\tilde{\mathcal{M}}_n = \left(\frac{q}{p}\right)^{X_n}.$$

$T \wedge N$ is a stopping time. By stopping theorem

$$\mathbf{E} \left[\left(\frac{q}{p} \right)^{X_{T \wedge N}} \right] = \mathbf{E} [\tilde{\mathcal{M}}_{T \wedge N}] = \mathbf{E} [\tilde{\mathcal{M}}_0] = \left(\frac{q}{p} \right)^x.$$

As $N \rightarrow \infty$, $X_{T \wedge N} \rightarrow X_T$, almost surely. By the bounded convergence theorem,

$$\begin{aligned} \lim_{N \rightarrow \infty} \mathbf{E} \left[\left(\frac{q}{p} \right)^{X_{T \wedge N}} \right] &= \mathbf{E} \left[\lim_{N \rightarrow \infty} \left(\frac{q}{p} \right)^{X_{T \wedge N}} \right] \\ &= \mathbf{E} \left[\left(\frac{q}{p} \right)^{X_T} \right] \end{aligned}$$

So

$$\mathbf{E} \left[\left(\frac{q}{p} \right)^{X_T} \right] = \left(\frac{q}{p} \right)^x.$$

But $X_T = 0$ or $X_T = b$. So,

$$\begin{aligned} \mathbf{E} \left[\left(\frac{q}{p} \right)^{X_T} \right] &= \left(\frac{q}{p} \right)^0 \mathbf{P} \{X_T = 0\} + \left(\frac{q}{p} \right)^b \mathbf{P} \{X_T = b\} \\ &= \left(\frac{q}{p} \right)^x \end{aligned}$$

Using the fact that $\mathbf{P} \{X_T = 0\} = 1 - \mathbf{P} \{X_T = b\}$ (This uses the fact that T is finite, almost surely.),

$$\left(\frac{q}{p} \right)^0 (1 - \mathbf{P} \{X_T = b\}) + \left(\frac{q}{p} \right)^b \mathbf{P} \{X_T = b\} = \left(\frac{q}{p} \right)^x$$

or

$$\mathbf{P} \{X_T = b\} = \frac{\left(\frac{q}{p} \right)^x - 1}{\left(\frac{q}{p} \right)^b - 1}.$$

$$\mu \mathbf{E} [T] = b \mathbf{P} \{X_T = b\} - x.$$

So

$$\mathbf{E} [T] = \frac{b \left(\frac{q}{p} \right)^x - 1}{\mu \left(\frac{q}{p} \right)^b - 1} - \frac{x}{\mu}.$$

Define $T_y = \inf\{n \geq 0 : X_n = y\}$. Then

$$\begin{aligned} P\{T_0 < T_b\} &= P\{X_{T_{0b}} = 0\} = 1 - \frac{\left(\frac{q}{p}\right)^x - 1}{\left(\frac{q}{p}\right)^b - 1} \\ &= \frac{\left(\frac{q}{p}\right)^b - \left(\frac{q}{p}\right)^x}{\left(\frac{q}{p}\right)^b - 1} \end{aligned}$$

Letting $b \rightarrow \infty$ yields

$$P\{T_0 < \infty\} = \lim_{b \rightarrow \infty} P\{X_{T_{0b}} = 0\} = \begin{cases} \left(\frac{q}{p}\right)^x & \text{if } q < p, \\ 0 & \text{if } q > p \end{cases}.$$

Exercise: Verify that $\mathbf{E}[T_0] < \infty$ if $q < p$.

Proof of Optional Stopping/Sampling Theorem: $T \leq K$ where K is chosen to be integer. Then,

$$\begin{aligned} \mathbf{E}[\mathcal{M}_T | \mathcal{F}_{K-1}] &= \mathbf{E}[(\mathcal{M}_T 1_{\{T=K\}} + \mathcal{M}_T 1_{\{T < K\}}) | \mathcal{F}_{K-1}] \\ &= \mathbf{E}[\mathcal{M}_T 1_{\{T=K\}} | \mathcal{F}_{K-1}] + \sum_{n=0}^{K-1} \mathbf{E}[\mathcal{M}_T 1_{\{T=n\}} | \mathcal{F}_{K-1}] \\ &\stackrel{(a)}{=} \mathbf{E}[\mathcal{M}_K 1_{\{T=K\}} | \mathcal{F}_{K-1}] + \sum_{n=0}^{K-1} \mathbf{E}[\mathcal{M}_n 1_{\{T=n\}} | \mathcal{F}_{K-1}] \\ &\stackrel{(b)}{=} \mathbf{E}[\mathcal{M}_K 1_{\{T \leq K-1\}^c} | \mathcal{F}_{K-1}] + \sum_{n=0}^{K-1} \mathcal{M}_n 1_{\{T=n\}} \\ &\stackrel{(c)}{=} \mathbf{E}[\mathcal{M}_K | \mathcal{F}_{K-1}] 1_{\{T \leq K-1\}^c} + \sum_{n=0}^{K-1} \mathcal{M}_n 1_{\{T=n\}} \\ &= \mathcal{M}_{K-1} 1_{\{T=K\}} + \sum_{n=0}^{K-1} \mathcal{M}_n 1_{\{T=n\}} \\ &= \mathcal{M}_{K-1} 1_{\{T \geq K-1\}} + \sum_{n=0}^{K-2} \mathcal{M}_n 1_{\{T=n\}} \end{aligned}$$

where (a) follows because $\mathcal{M}_T 1_{\{T=n\}} = \mathcal{M}_n 1_{\{T=n\}} \forall n$, (b) follows from the fact that for $n \leq K-1$, $\mathcal{M}_n 1_{\{T=n\}} \in \mathcal{F}_n \subset \mathcal{F}_{K-1}$, (c) follows because $\{T \leq K-1\} = \cup_{n=0}^{K-1} \{T=n\} \in \mathcal{F}_{K-1}$ and thus, $\{T \leq K-1\}^c \in \mathcal{F}_{K-1}$.

$$\begin{aligned}
 \mathbf{E}[\mathcal{M}_T | \mathcal{F}_{K-2}] &= \mathbf{E}[\mathbf{E}[\mathcal{M}_T | \mathcal{F}_{K-1}] | \mathcal{F}_{K-2}] \\
 &= \mathbf{E}\left[\left(\mathcal{M}_{K-1} 1_{\{T \geq K-1\}} + \sum_{n=0}^{K-2} \mathcal{M}_n 1_{\{T=n\}}\right) | \mathcal{F}_{K-2}\right] \\
 &= \mathbf{E}[\mathcal{M}_{K-1} 1_{\{T \geq K-1\}} | \mathcal{F}_{K-2}] + \sum_{n=0}^{K-2} \mathcal{M}_n 1_{\{T=n\}} \\
 &= \mathbf{E}[\mathcal{M}_{K-1} | \mathcal{F}_{K-2}] 1_{\{T \geq K-1\}} + \sum_{n=0}^{K-2} \mathcal{M}_n 1_{\{T=n\}} \\
 &= \mathcal{M}_{K-2} 1_{\{T \geq K-2\}} + \sum_{n=0}^{K-3} \mathcal{M}_n 1_{\{T=n\}}.
 \end{aligned}$$

Proceeding in this way,

$$\mathbf{E}[\mathcal{M}_T | \mathcal{F}_0] = \mathcal{M}_0 1_{\{T \geq 0\}} = \mathcal{M}_0$$

which implies that

$$\mathbf{E}[\mathcal{M}_T] = \mathbf{E}[\mathcal{M}_0].$$

□