

X is a random Variable:

$$X: \Omega \rightarrow \mathfrak{R}$$

$$X^{-1}((a, b])$$

$$= \{\omega \in \Omega; X(\omega) \in (a, b]\}$$

$$X^{-1}((a, b]) \in \mathcal{F} \text{ for all } (a, b] : -\infty < a < b < \infty$$

Distribution function for X : $F: \mathfrak{R} \rightarrow [0, 1]$

$$F(X) = P(X \leq x) = P(X^{-1}(-\infty, x])$$

$\sigma(X)$ = Smallest σ -algebra containing $X^{-1}((a, b]) ; -\infty < a < b < \infty$

Let $A \in \sigma(X); A = X^{-1}(B)$, B is Borel set

Definition: Borel set is smallest σ -algebra in $\mathfrak{R}$ containing all intervals of form $(a, b] ; -\infty < a < b < \infty$

$$1_A(\omega) = 1_{\{X \in B\}}(\omega)$$

$\sigma(X)$ = events that you can distinguish knowing X.

$X = (X_0, X_1, \dots, X_n)$ is a random vector. $X_i; \Omega \rightarrow \mathfrak{R}$ is a random variable $i=0, 1, \dots, n$.

$\sigma(X)$ is smallest σ -algebra containing all events of form $X_i^{-1}((a, b])$, $i=0, 1, \dots, n, -\infty < a < b < \infty$

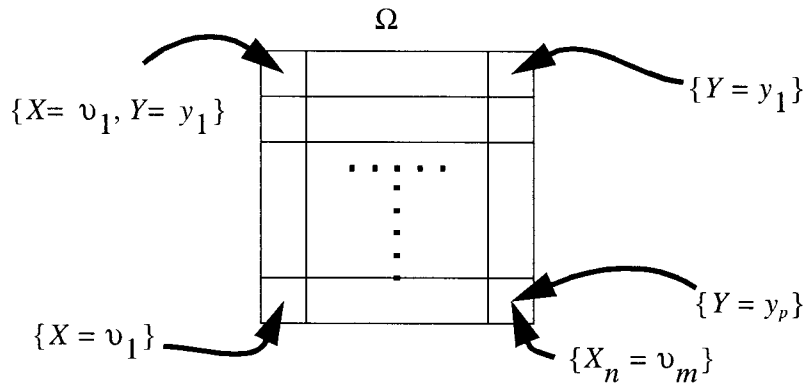
Suppose Y is another random variable and $E[|Y|] < \infty$

$E[Y|\sigma(X)] = E[Y|X] = \Phi(X)$ where $\Phi: \mathfrak{R}^{n+1} \rightarrow \mathfrak{R}$ is a measurable function; For any

$A \in \sigma(X)$ we have: $E[1_A E[Y|X]] = E[Y 1_A]$

Special case: X and Y take only finitely many values, then we partition Ω according to inverse image of X, as is illustrated in the figure below:





$$E[Y|X] = \Phi(X), \text{ on } \{X = v_1\}, \Phi(X) = C_1$$

$$C_1 \cdot P(A_1) = E[Y 1_{A_1}] \quad A_1 = \{X = v_1\}$$

$$C_1 = \frac{E[Y 1_{A_1}]}{P(A_1)} = \sum_{i=1}^p \frac{y_i P(Y = y_i, X = v_1)}{P(X = v_1)} = \sum_{i=1}^p P(Y = y_i | X = v_1) y_i = E[Y | X = v_1]$$

MARTINGALE
 $M = \{M_n, \mathcal{F}_n, n = 0, 1, \dots\}$ generated by $\mathcal{F}_n$ are observed random variables in a sequence. (information increases in time)

$$E[M_{n+1} | \mathcal{F}_n] = M_n = E[M_n | \mathcal{F}_n], n=0, 1, 2, \dots$$

$$\Rightarrow E[M_{n+1} - M_n | \mathcal{F}_n] = 0 \quad (\text{fair game}) \quad M_{n+1} - M_n \text{ are martingale differences.}$$

$$E[M_{n+1} 1_\Omega] = E[M_n 1_\Omega] \Rightarrow E[M_{n+1}] = E[M_n] \Rightarrow E[M_n] = E[M_0], \forall n$$

(often $\mathcal{F}_n = \sigma\{M_0, M_1, \dots, M_n\}$)

Example 1: Random Walk:

Suppose we have a sequence of random variables $\{\xi_i\}_{i=1}^\infty$ which are independent and identically distributed, $E[|\xi_i|] < \infty, \forall i, \mu = E[\xi_i], \forall i$

$$X_n = x + \sum_{i=1}^n \xi_i$$

$n = 0, 1, 2, \dots, x \in \mathbb{R}, \sum_{i=1}^0 \xi_i = 0, \{X_n\}_{n=0}^\infty$ is random walk.

$$\text{Let } M_n = X_n - n\mu, \mathcal{F}_n = \sigma\{M_0, M_1, \dots, M_n\} = \sigma\{X_0, X_1, \dots, X_n\} = \sigma\{\xi_1, \dots, \xi_n\}$$

$$E[|M_n|] \leq E[|X_n|] + n|\mu| \leq |x| + \sum_{i=1}^n E[|\xi_i|] + n|\mu| < \infty$$

$$\begin{aligned} \text{and also: } E[M_{n+1} | \mathcal{F}_n] &= E[X_{n+1} - (n+1)\mu | \mathcal{F}_n] = E[X_{n+1} | \mathcal{F}_n] - (n+1)\mu \\ &= E[X_n + \xi_{n+1} | \mathcal{F}_n] - (n+1)\mu = E[X_n | \mathcal{F}_n] + E[\xi_{n+1} | \mathcal{F}_n] - (n+1)\mu \\ &= X_n + E[\xi_{n+1}] - (n+1)\mu = X_n + \mu - (n+1)\mu = X_n - n\mu = M_n \end{aligned}$$

(because $\xi_{n+1} \perp X_n, X_{n-1}, \dots, X_0 \Rightarrow \xi_{n+1} \perp \mathcal{F}_n$)

$M_{n+1} - M_n = X_{n+1} - X_n - \mu = \xi_{n+1} - \mu$, ($\xi_{n+1} - \mu$ are iid with mean zero)

special case:

$\{\xi_i\}_{i=1}^{\infty}$ is iid:

$P(\xi_i = 1) = p, P(\xi_i = -1) = q = 1 - p, 0 < p < 1, p \neq q$ non symmetric case

$$M_n = (q/p)^{X_n}, n = 0, 1, 2, \dots, X_n = x + \sum_{i=1}^n \xi_i \text{ x is integer, } \mathcal{F}_n = \sigma\{X_0, X_1, \dots, X_n\}$$

1) $M_n \in \mathcal{F}_n$ since $X_n \in \mathcal{F}_n$

2) $E[(q/p)^{X_n} < \infty]$ since X_n is bounded for each n.

$$\begin{aligned} 3) E[M_{n+1} | \mathcal{F}_n] &= E[(q/p)^{X_{n+1}} | \mathcal{F}_n] = E[(q/p)^{X_n + \xi_{n+1}} | \mathcal{F}_n] \\ &= E[(q/p)^{X_n} \cdot (q/p)^{\xi_{n+1}} | \mathcal{F}_n] = (q/p)^{X_n} E[(q/p)^{\xi_{n+1}} | \mathcal{F}_n] \\ &= (q/p)^{X_n} E[(q/p)^{\xi_{n+1}}] = (q/p)^{X_n} [(q/p)^1 \cdot p + (q/p)^{-1} \cdot q] = (q/p)^{X_n} = M_n. \end{aligned}$$

Example 2): Branching Process

$$X_0 = 1, X_{n+1} = \text{size of population at generation } n+1 = \sum_{i=1}^{X_n} \xi_i^{(n+1)} \quad (\xi_i^{(n+1)} = \# \text{ of offspring of individual } i \text{ in generation } n)$$

$$\{\xi_i^{(n+1)}, i = 1, 2, \dots\} \text{ iid, } E[\xi_i^{(n+1)}] = \mu < \infty$$

Assume these sequences are mutually indept as n varies.

$$M_n = \frac{X_n}{\mu} \quad \mathcal{F}_n = \sigma\{X_0, X_1, \dots, X_n\}$$

$$1) M_n \in \mathcal{F}_n \quad n=0,1,2,\dots$$

$$2) E[X_{n+1}] = E\left[\sum_{i=1}^{X_n} \xi_i^{(n+1)}\right] = \sum_{j=0}^{\infty} E\left[\sum_{i=1}^{X_n} \xi_i^{(n+1)} \cdot 1_{\{X_n=j\}}\right]$$

$$= \sum_{j=0}^{\infty} E\left[\sum_{i=1}^j \xi_i^{(n+1)} \cdot 1_{\{X_n=j\}}\right] = \sum_{j=0}^{\infty} \sum_{i=1}^j E[\xi_i^{(n+1)}] \cdot E[1_{\{X_n=j\}}]$$

← using independence of $\xi_i^{(n+1)}$ from X_n .

$$= \sum_{j=1}^{\infty} \mu \cdot 1_{\{X_n=j\}} = \mu E[X_n] \text{ use induction to prove } E[X_n] \text{ is finite hence } E[M_n] \text{ is finite for all } n.$$

$$3) E[M_{n+1} | \mathcal{F}_n] = \frac{1}{\mu_{n+1}} E[X_{n+1} | \mathcal{F}_n] = \frac{1}{\mu_{n+1}} E\left[\sum_{i=1}^{X_n} \xi_i^{(n+1)} \middle| \mathcal{F}_n\right]$$

$$= \frac{1}{\mu_{n+1}} \sum_{i=1}^{X_n} E[\xi_i^{(n+1)} | \mathcal{F}_n] = \frac{1}{\mu_{n+1}} X_n \mu = \frac{X_n}{\mu_n} = M_n$$

Example:

$\{\xi_i\}$ are iid with mean μ and variance $\sigma^2 < \infty$.

$$X_n = \sum_{i=1}^n \xi_i, X_0 = 0$$

We know $M_n = X_n - n\mu$ defines martingale with respect to $\mathcal{F}_n = \{X_0, X_1, \dots, X_n\}$

we claim that $N_n = M_n^2 - n\sigma^2$ is a martingale.

Proof:

$$1) N_n \in \mathcal{F}_n \quad \forall n$$

$$2) E[|N_n|] \leq E[M_n^2] + n\sigma^2 = \sum_{i=1}^n E[(\xi_i - \mu)^2] + n\sigma^2 = 2n\sigma^2 < \infty$$

$$\begin{aligned}
3) E[N_{n+1} | \mathcal{F}_n] &= E[M_{n+1}^2 - (n+1)\sigma^2 | \mathcal{F}_n] = E[(M_n + \xi_{n+1} - \mu)^2 - (n+1)\sigma^2 | \mathcal{F}_n] \\
&= E[M_n^2 + 2M_n(\xi_{n+1} - \mu) + (\xi_{n+1} - \mu)^2 | \mathcal{F}_n] - (n+1)\sigma^2 \\
&= M_n^2 + 2M_n E[\xi_{n+1} - \mu] + E[(\xi_{n+1} - \mu)^2] - (n+1)\sigma^2 \\
&= M_n^2 + \sigma^2 - (n+1)\sigma^2 = M_n^2 - n\sigma^2 = N_n.
\end{aligned}$$