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Lecture 14

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H.W. # 4 is on web, Due June 1

Problem 1. Computation

Problem 2. Modeling

Problem 3. Modeling, Stationary Distribution

Problem 4. Queueing system,
Stationary Distribution.

Continuous time Markov Chains (cont.).

From modeling, identify Q matrix of
infinitesimal rates:

$$Q = (Q_{ij})$$

$$(i) -Q_{ii} \geq 0 \quad \forall i$$

$$(ii) Q_{ij} \geq 0 \quad \forall i \neq j$$

$$(iii) \sum_{j \neq i} Q_{ij} < \infty \quad \forall i$$

$$(iv) Q_{ii} = - \sum_{j \neq i} Q_{ij} \quad \forall i$$

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Given such a matrix Q , can define holding time parameters

$$\lambda_i = -Q_{ii} \quad \forall i, \quad (\lambda_i = 0 \Rightarrow i \text{ is absorbing})$$

$$\tilde{P}_{ij} = \frac{Q_{ij}}{\lambda_i} \quad j \neq i \text{ and } \lambda_i \neq 0$$

$$\tilde{P}_{ii} = \begin{cases} 0 & \text{if } \lambda_i \neq 0 \\ 1 & \text{if } \lambda_i = 0 \end{cases}$$

Given a Q -matrix, can always construct a right continuous

Markov chain on $S \cup \{\infty\}$ cemetery with infinitesimal rate matrix Q .

Can construct this by using exponential holding times in successive states $S_1, S_2, \dots$ with transition probabilities given by $\tilde{P}$. If $T_1, T_2, \dots$ are successive holding times,

Then

$$\rho = \sum_{i=1}^{\infty} T_i \text{ is "explosion time"}$$

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if $P < \infty$ then define $X(t) = \infty$

for all $t \geq P$

(minimal M.C. associated with Q).

Explosion will not occur in finite time if:

(i) S is finite

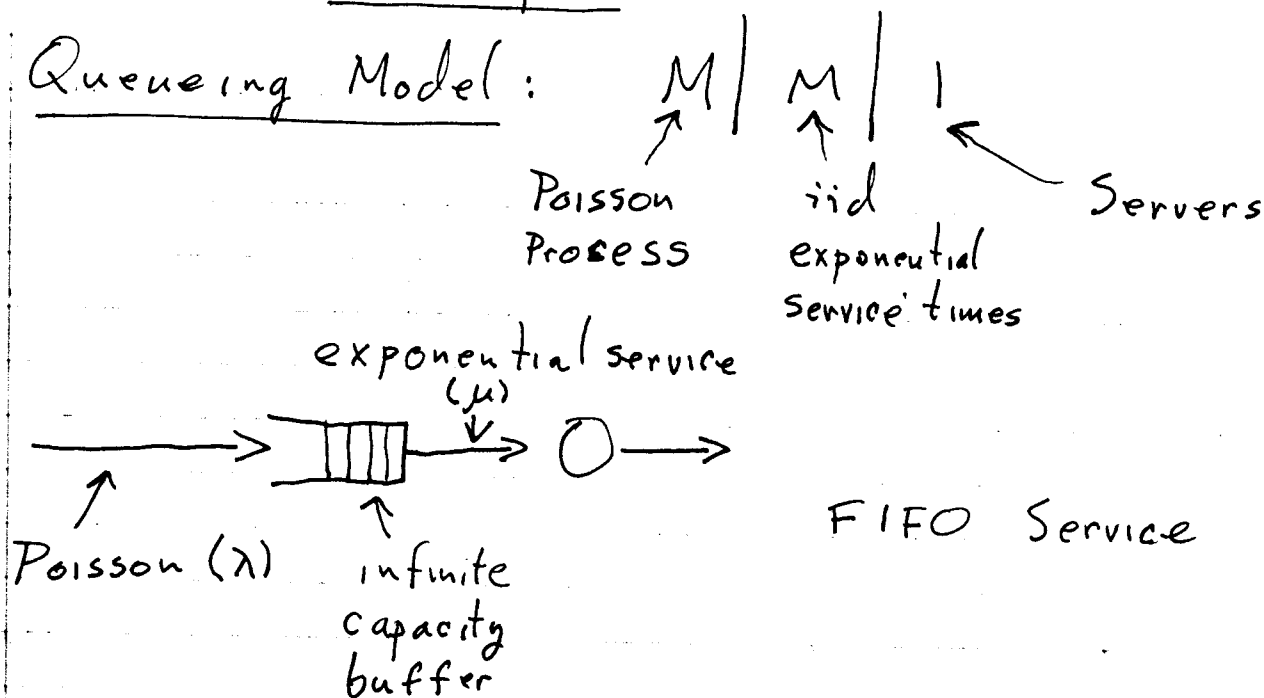
(ii) $\{\lambda_i\}_{i=1}^{\infty}$ is bounded

(iii) M.C. is recurrent (i.e. M.C.

associated with $\tilde{P}$ is recurrent)

Example

Queueing Model:



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$X(t)$ = # of jobs in the system
at time t

$$S = \mathbb{N}$$

$$T = [0, \infty)$$

$$P(X(t+h) = i+1 \mid X(t) = i) = \lambda h + o(h)$$

$$P(X(t+h) = i-1 \mid X(t) = i) = \mu h + o(h) \quad i > 0$$

$$\rightarrow Q_{i,i+1} = \lambda \quad i = 0, 1, 2, \dots$$

$$Q_{i,i-1} = \mu \quad i = 1, 2, 3, \dots$$

$$Q_{i,i} = \begin{cases} -(\lambda + \mu), & i = 1, 2, \dots \\ -\lambda, & i = 0 \end{cases}$$

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & \dots & 0 \\ \mu & -(\lambda + \mu) & \lambda & 0 & \dots & 0 \\ & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

LONG RUN BEHAVIOR

Definition: M.C. is irreducible if
for each $i \neq j$,

$$P_{ij}(t) > 0 \text{ for some } t > 0$$

(Equivalent to irreducibility of embedded
discrete time M.C.)

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Lemma

M.C. is irreducible iff for each $i \neq j$, there are distinct states $i_0 = i, i_1, i_2, \dots, i_n = j$

s.t. $Q_{i_0 i_1} Q_{i_1 i_2} \dots Q_{i_{n-1} i_n} > 0$

Do not need to worry about periodicity of continuous time M.C.'s

Recurrence & Transience

are determined by embedded discrete time M.C. (with transition matrix $\bar{P}$)

i is transient: $P_i(\{t \geq 0 : X_t = i\} \text{ is unbounded}) = 0$

i is Recurrent: $P_i(\{t \geq 0 : X_t = i\} \text{ is unbounded}) = 1$

Let $T_i = \inf \{t \geq T_i : X(t) = i\}$, $T_i =$ first time X leaves i

i transient $\Leftrightarrow -Q_{ii} > 0$ & $P_i(T_i < \infty) < 1$

i recurrent $\Leftrightarrow$ either $Q_{ii} = 0$ or $P_i(T_i < \infty) = 1$

$\rightarrow$ If Q is irreducible and recurrent then there is non-explosion.

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i is Positive Recurrent

if $Q_{ii} = 0$ (i.e. only one state)
 $\nearrow$ when irreducible

or $E_i[T_i] < \infty$

(not determined by embedded discrete time M.C.).

Stationary Distribution (invariant distribution):

is a probability distribution π on S :

$$\pi Q = 0$$

Theorem: Suppose M.C. is irreducible with Q -matrix Q then all states are positive recurrent iff Q has a stationary dist'n and Q is non-explosive. In this case,

$$\lim_{t \rightarrow \infty} P_{ij}(t) = \pi_j \text{ for all } i, j$$

Remark: Could show non-explosiveness by showing recurrence (e.g. for embedded discrete time M.C.)

Aside:

Q is a generator. Formally:

$$P(t) = (P_{ij}(t))$$

$$= e^{Qt} = \sum_{n=0}^{\infty} \frac{Q^n t^n}{n!} = I + \sum_{n=1}^{\infty} \frac{Q^n t^n}{n!}$$

$$\pi P(t) = \pi \quad \text{if} \quad \pi Q = 0 \quad (\text{rigorous if } S \text{ is finite}).$$

Detailed Balance

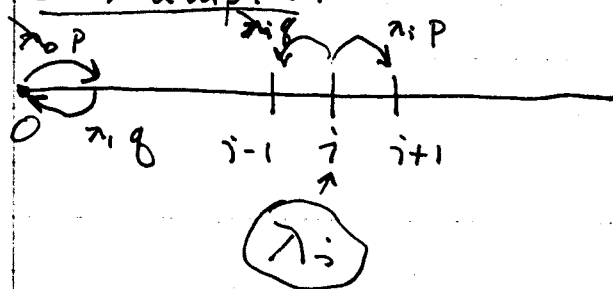
Probability dist'n π :

$$\boxed{\pi_i Q_{ij} = \pi_j Q_{ji} \quad \forall i, j}$$

$$\Rightarrow \sum_i \pi_i Q_{ij} = \sum_i \pi_i Q_{ji} = 0$$

$$\Rightarrow (\pi Q)_j = 0 \quad \forall j \quad \text{So } \pi \text{ is a stationary distribution}$$

Example:



$$0 < p < 1$$

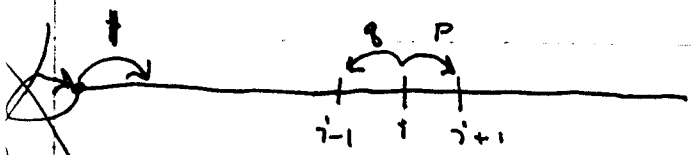
$$q = 1 - p$$

embedded discrete time M.C.

From prior analysis:
Transient

iff $p > q$

Recurrent iff $p \leq q$



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Stationary Dist'n

Seek probability distribution π :

detailed Balance holds -

$$\pi_i Q_{i,i+1} = \pi_{i+1} Q_{i+1,i} \quad \text{①} \quad \text{"balanced flow"}$$

$$\pi_i \lambda_i p = \lambda_{i+1} q \pi_{i+1} \quad i=0,1,2,\dots$$

$$\pi_{i+1} = \frac{\lambda_i}{\lambda_{i+1}} \frac{p}{q} \pi_i \quad i=0,1,2,\dots$$

$$\Leftrightarrow \pi_i = \frac{\lambda_0}{\lambda_i} \left(\frac{p}{q}\right)^i \pi_0 \quad i=1,2,\dots$$

Case 1: $\lambda_i = \lambda > 0 \quad \forall i$

π det. a stationary distribution iff. $p < q$

In this case

$$\sum_{i=0}^{\infty} \pi_i = 1 \Rightarrow \sum_{i=0}^{\infty} \left(\frac{p}{q}\right)^i \pi_0 = 1$$

$$\Rightarrow \pi_0 = 1 - p/q$$

$$\pi_i = \left(\frac{p}{q}\right)^i (1 - p/q) \quad \forall i$$

For $p < q$, have irreducible, recurrent

M.C., and stationary dist'n π . So

M.C. is pos recurrent and

$$\lim_{t \rightarrow \infty} P_{ij}(t) = \pi_j \quad \forall i, j$$

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Case 2: $\lambda_i = 2^i$

$$\pi_i = \frac{1}{2^i} \left(\frac{p}{q}\right)^i \quad \pi_0, \quad i = 1, 2, \dots$$

Summable iff $\frac{p}{2q} < 1 \Leftrightarrow \frac{p}{q} < 2$

$\Rightarrow$ Stationary dist'n iff $\frac{p}{q} < 2$

if $\frac{p}{q} \leq 1$, have recurrence and so positive recurrence.

If $1 < \frac{p}{q} < 2$, have transience and so not pos. recurrence and so by theorem, we must have explosion.

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