

Continuous Time Markov Chains

Definition : A continuous time Markov chain is a stochastic process $\{X(t), t \in [0, \infty)\}$ taking values in a discrete (finite or countably infinite set) S such that :

$$\begin{aligned} P(X(t) = j \mid X(s) = i, X(s_1) = i_1, \dots, X(s_n) = i_n) \\ = P(X(t) = j \mid X(s) = i), \quad \forall 0 \leq s_1 < s_2 < \dots < s_n < s < t \\ i, j, i_1, \dots, i_n \in S \end{aligned}$$

Informally : $P(X(t) = j \mid X(u) : u \leq s) = P(X(t) = j \mid X(s))$

We will restrict to time homogeneous Markov chains.

$$\begin{aligned} P(X(t+u) = j \mid X(t) = i) &= P(X(u) = j \mid X(0) = i), \quad \forall 0 \leq u, t < \infty. \\ &= P_{ij}(u), \quad u \geq 0. \end{aligned}$$

Transition Probabilities

① $P_{ij}(u) \geq 0$ for all $i, j \in S$, $u \geq 0$

② $\sum_{j \in S} P_{ij}(u) = 1$, $\forall i \in S$, $u \geq 0$

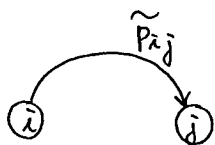
③ $P_{ii}(t) \rightarrow 1$ as $t \rightarrow 0$ for all $i \in S$

④ Chapman - Kolmogorov equations.

$$\begin{aligned} P_{ij}(t+s) &= P(X(t+s) = j \mid X(0) = i), \quad 0 < s, t \\ &= \sum_{k \in S} P(X(t+s) = j, X(s) = k \mid X(0) = i) \\ &= \sum_{k \in S} P(X(t+s) = j \mid X(s) = k, X(0) = i) \cdot P(X(s) = k \mid X(0) = i) \\ &= \sum_{k \in S} P_{kj}(t) P_{ik}(s) \\ &= (P(s) P(t))_{ij} \end{aligned}$$

Two different ways of thinking about continuous time Markov chains

- (1) - Have a transition matrix $\tilde{P}$ for a discrete time Markov chain on state space S : $\tilde{P}_{ii} = 0$ for all $i \in S$



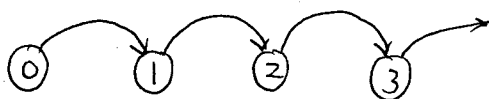
$\tilde{P}_{ij}$ = probability of transitioning from i to j when eventually leave i ($j \neq i$)

- For each state i , there is a parameter $\lambda_i > 0$ for an exponential r.v. - holding time in state i is exponential with parameter λ_i .

Dynamics of Continuous Time Markov Chain

- ① Start in some state i
- ② Wait in i for an exponentially distributed amount of time with parameter λ_i (holding time)
- ③ When holding time is up, move to a new state j with probability $\tilde{P}_{ij}$
- ④ In new state j , reset exponential alarm clock and repeat as above

Warning : It is possible that Markov chain could run off to infinity in finite time (explosion).



$$\tilde{P}_{i, i+1} = 1, \quad i = 0, 1, 2, \dots$$

mean holding time in i is $\frac{1}{\lambda_i}$

time to reach ∞ (explode) $\rightarrow T = \sum_{i=0}^{\infty} T_i$, T_i = holding time in state $i \sim \exp(\lambda_i)$

$$E[T] = \sum_{i=0}^{\infty} E[T_i] = \sum_{i=0}^{\infty} \frac{1}{\lambda_i}$$

If $\sum_{i=0}^{\infty} \lambda_i < \infty$, then $E[T] < \infty \Rightarrow T < \infty$ a.s.

If S is finite, explosion is impossible.

If $\lambda_i \leq c$ for all i , avoid explosion

(2) Infinitesimal description

$$P_{ij}(h) = \underbrace{Q_{ij}}_{\geq 0} h + o(h), \quad j \neq i \quad \left(\frac{o(h)}{h} \rightarrow 0 \text{ as } h \rightarrow 0 \right)$$

$$\begin{matrix} \textcircled{i} & \textcircled{j} \\ \lambda_i h & \tilde{P}_{ij} \end{matrix}$$

$$P_{ii}(h) = 1 + Q_{ii} h + o(h), \quad Q_{ii} = - \sum_{j \neq i} Q_{ij}$$

$$Q = \begin{pmatrix} Q_{00} = - \sum_{j \neq 0} Q_{0j} & Q_{01} & \dots \\ & & \end{pmatrix}, \quad \text{row sums are } 0.$$

↑
negative entries

$$\left. \frac{dP(t)}{dt} \right|_{t=0} = Q$$

$$P(0) = I$$

Chapman - Kolmogorov : $P(t)P(h) = P(h)P(t) = P(t+h)$

$$\Rightarrow \underbrace{P(t)}_{\text{forward equation}} Q = Q \underbrace{P(t)}_{\text{backward equation}} = \underbrace{\frac{dP(t)}{dt}}_{\substack{\text{Kolmogorov equation} \\ \text{or} \\ \text{Master equation}}}$$

Connection between (1) & (2) :

$$Q_{ij} = \lambda_i \tilde{P}_{ij}, \quad j \neq i$$

$$-Q_{ii} = \sum_{j \neq i} Q_{ij} = \lambda_i$$

Examples

① Poisson Process ($\lambda > 0$)

$$S = \mathbb{N} = \{0, 1, 2, \dots\}$$

$\lambda_i = \lambda$ — exponential parameter same for all i

Start at zero usually.

$$\tilde{P}_{i, i+1} = 1, \quad i = 0, 1, 2, \dots$$

$$Q = \begin{pmatrix} -\lambda & \lambda & & \\ & -\lambda & \lambda & \\ & & \ddots & \ddots \\ \bigcirc & & & \bigcirc \end{pmatrix}$$

$$Q_{i, i+1} = \lambda$$

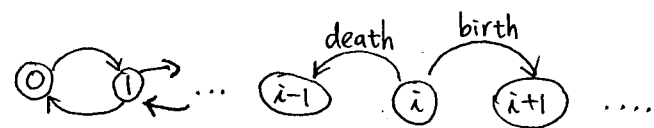
$X(t) = \#$ of rate events that occurred up to t

Applications: radioactive decay, supernova observations, telephone calls.

② Birth-death Processes

$$S = \mathbb{N}.$$

Applications: Epidemiology, operations management — queueing model.



$X(t) = \#$ of individuals at time t .

$\tilde{\lambda}_i =$ rate of birth of new individual when there are i individuals

$\tilde{\mu}_i =$ rate of death when there are i individuals

$$\exp(\tilde{\lambda}_i) \wedge \exp(\tilde{\mu}_i) \xrightarrow{\text{distribution}} \exp(\tilde{\lambda}_i + \tilde{\mu}_i)$$

$$\lambda_i = \text{rate of leaving } i = \tilde{\lambda}_i + \tilde{\mu}_i$$

$$\tilde{P}_{i, i+1} = \frac{\tilde{\lambda}_i}{\tilde{\lambda}_i + \tilde{\mu}_i}, \quad \tilde{P}_{i, i-1} = \frac{\tilde{\mu}_i}{\tilde{\lambda}_i + \tilde{\mu}_i}, \quad i \geq 1$$

$$Q = \begin{pmatrix} -\tilde{\lambda}_0 & \tilde{\lambda}_0 & 0 & \dots \\ \tilde{\mu}_1 & -(\tilde{\lambda}_1 + \tilde{\mu}_1) & \tilde{\lambda}_1 & \\ 0 & \tilde{\mu}_2 & -(\tilde{\lambda}_2 + \tilde{\mu}_2) & \tilde{\lambda}_2 & 0 & \dots \\ \vdots & & & & & \end{pmatrix}$$

Pure birth : $\tilde{\mu}_i = 0, \forall i$

Pure death : $\tilde{\lambda}_i = 0, \forall i$

Yule : $\tilde{\lambda}_i = i \cdot \lambda, \forall i$

Genetic Example

Population of N individuals

Two types of individuals: a, A .

$X(t)$ = # of individuals of type a at time t .

In any small time interval of length h , probability system changes is $\lambda h + o(h)$

Given change is to occur at $t+h$, and $X(t) = j$,
 pick individual at random from population to die;
 pick individual at random to give birth:

r_1 = probability an a -type gives birth to A -type.

r_2 = probability an A -type gives birth to a -type.

probability of new birth of type a is $(\frac{j}{N})(1-r_1) + (1-\frac{j}{N}) \cdot r_2$

$$\tilde{p}_{j,j+1} = (1 - \frac{j}{N}) \cdot \left[\frac{j}{N}(1-r_1) + (1 - \frac{j}{N})r_2 \right], \quad j = 0, 1, \dots, N-1$$

$$\tilde{p}_{j,j-1} = \left(\frac{j}{N} \right) \cdot \left[\frac{j}{N} \cdot r_1 + (1 - \frac{j}{N})(1-r_2) \right], \quad j = 1, 2, \dots, N.$$

kill off
a-type

$$\lambda_j = \lambda$$

$$\text{fraction of } a\text{'s in population} = \frac{X(t)}{N}$$