

MATH 280C**HOMEWORK #3, DUE APRIL 27, 2011.**

Please hand in the starred problems. In the following, all random variables are defined on the probability space $(\Omega, \mathcal{F}, P)$.

1. A Poisson process with parameter $\lambda > 0$ is a stochastic process $N = \{N(t), t \geq 0\}$ such that

- $N(0) = 0$,
- for any positive integer n and times $0 \leq t_1 < t_2 < \dots < t_n$, $(N(t_1), N(t_2) - N(t_1), \dots, N(t_n) - N(t_{n-1}))$ are independent, and
- for each $0 \leq s < t < \infty$, $N(t) - N(s)$ is a Poisson random variable with parameter $\lambda(t - s)$.

Note: N is a process with stationary independent (Poisson) increments.

For each $t \geq 0$, let $\mathcal{H}_t = \sigma\{N(s) : 0 \leq s \leq t\}$, the smallest σ -algebra with respect to which $N(s)$ is measurable for each $0 \leq s \leq t$. Suppose that $(\Omega, \mathcal{F}, P)$ is complete. Let $\mathcal{N}$ denote the P -null sets in $\mathcal{F}$. For each $t \geq 0$, let $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{N}$, the smallest σ -algebra containing $\mathcal{H}_t$ and $\mathcal{N}$ and let $\mathcal{F}_t = \mathcal{G}_{t+}$.

(a) Prove that $\{N(t), \mathcal{F}_t, t \geq 0\}$ is a submartingale. Hint: first establish this with $\mathcal{G}_t$ in place of $\mathcal{F}_t$.

(b) Use the result of (a) to conclude that there is a modification $\tilde{N}$ of N such that the paths of $\tilde{N}$ are right continuous with finite left limits (càdlàg) and $\{\tilde{N}_t, \mathcal{F}_t, t \geq 0\}$ is a submartingale where $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ satisfies the usual conditions. (For this reason, when one uses a Poisson process, one usually assumes that it has càdlàg paths.)

2. Suppose that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ is a filtered probability space. Suppose that S and T are stopping times on this space. Prove that

- (a) $S \wedge T$ and $S \vee T$ are stopping times.
 (b) if $S \leq T$, then $\mathcal{F}_S \subset \mathcal{F}_T$.

Now suppose that $\{T_n\}_{n=0}^\infty$ is a sequence of stopping times and that $\{\mathcal{F}_t, t \geq 0\}$ is right continuous. Prove that

- (c) $\inf_n T_n$, $\sup_n T_n$, $\liminf_n T_n$ and $\limsup_n T_n$ are also stopping times.
 (d) if $T_{n+1} \leq T_n$ for all n , then for $T = \inf_n T_n$, $\mathcal{F}_T = \cap_n \mathcal{F}_{T_n}$.

3*. Suppose that $\{X_t, t \geq 0\}$ is a continuous adapted process defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ where $\{\mathcal{F}_t\}$ is right continuous. Let A be a closed set in $(-\infty, \infty)$. Prove that

$$T_A = \inf\{t \geq 0 : X_t \in A\}$$

is a stopping time. *Hint:* for each positive integer n , $A^n = \{x : \text{dist}(x, A) < 1/n\}$ is open.

4*. A standard one-dimensional Brownian motion is a real-valued stochastic process $\{B_t, t \geq 0\}$ such that

- $B(0) = 0$,
- for any positive integer n and times $0 \leq t_1 < t_2 < \dots < t_n$, $\{B(t_1), B(t_2) - B(t_1), \dots, B(t_n) - B(t_{n-1})\}$ are independent,
- for each $0 \leq s < t < \infty$, $B(t) - B(s)$ has a normal distribution with mean zero and variance $t - s$,
- the paths of B are continuous.

(This is another process with stationary, independent increments).

For each $t \geq 0$, let $\mathcal{H}_t = \sigma\{B(s) : 0 \leq s \leq t\}$, the smallest σ -algebra with respect to which $B(s)$ is measurable for each $0 \leq s \leq t$. Suppose that $(\Omega, \mathcal{F}, P)$ is complete and let $\mathcal{N}$ denote the P -null sets in $\mathcal{F}$. Let $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{N}$ and let $\mathcal{F}_t = \mathcal{G}_{t+}$ for all $t \geq 0$.

(a) Prove that $\{B(t), \mathcal{F}_t, t \geq 0\}$ and $\{B_t^2 - t, \mathcal{F}_t, t \geq 0\}$ are martingales. Hint: first prove this with $\mathcal{G}_t$ in place of $\mathcal{F}_t$.

(b) Let $T = \inf\{t \geq 0 : B_t > 1\}$. Is T a stopping time relative to $\{\mathcal{F}_t, t \geq 0\}$? Make sure to justify your answer.

(c) Fix $a < 0 < b$ and let $S = \inf\{s \geq 0 : B_s \leq a \text{ or } B_s \geq b\}$. Is S a stopping time relative to $\{\mathcal{F}_t, t \geq 0\}$? Use Doob's stopping theorem to show that $E[S] < \infty$, $E[B_S] = 0$ and to compute the probability that B hits a before b . You should also be able to compute $E[S]$.

(d) (Extra credit) Prove that $\{M_t = \exp(cB_t - \frac{1}{2}c^2t), \mathcal{F}_t, t \geq 0\}$ is a martingale for any real number c . (You may use results about the moment generating function for the normal distribution.) Prove that $\lim_{t \rightarrow \infty} M_t$ exists a.s. Can you identify this limit?