

MATH 280C**HOMEWORK #2, DUE APRIL 13, 2011.**

Please hand in the starred problems. In the following, all random variables are defined on the probability space $(\Omega, \mathcal{F}, P)$.

1. Suppose that $\{X_n, \mathcal{F}_n, n = 0, 1, 2, \dots\}$ is a submartingale. Prove that for any $\lambda > 0$ and $n \geq 0$,

$$\lambda P\left(\min_{0 \leq i \leq n} X_i \leq -\lambda\right) \leq E(X_n^+) - E(X_0).$$

Combine this with the version of Doob's inequality proved in class (involving the max) to prove that

$$\lambda P\left(\max_{k=0}^n |X_k| \geq \lambda\right) \leq 2E[X_n^+] - E[X_0].$$

2*. Suppose that $\{\mathcal{F}_t, t \geq 0\}$ is a filtration. For each $t \geq 0$, let $\mathcal{G}_t = \mathcal{F}_{t+} = \cap_{s>t} \mathcal{F}_s$. Prove that $\{\mathcal{G}_t, t \geq 0\}$ is a right continuous filtration.

3*. Suppose that $(\Omega, \mathcal{F}, P)$ is a complete probability space and suppose that $\mathcal{G} \subset \mathcal{F}$ is a σ -algebra that contains all of the P -null sets in $\mathcal{F}$. Suppose that X is an $\mathcal{F}$ -measurable random variable and Y is a $\mathcal{G}$ -measurable random variable such that $X = Y$ a.s. (under P). Prove that X is also $\mathcal{G}$ -measurable.

4. Suppose that $(\Omega, \mathcal{F}, P)$ is a complete probability space and that $\{\mathcal{F}_t, t \geq 0\}$ is a filtration where $\mathcal{F}_t$ contains all of the P -null sets for each $t \geq 0$. Prove that the process $\{Y_t, t \geq 0\}$ defined in class is (surely) right continuous with finite left limits. (A sketch of this was done in class. This problem is just to remind you to make sure you understand that proof.)

5. Suppose that $(\Omega, \mathcal{F}, P)$ is a complete probability space and $\{\mathcal{F}_t, t \geq 0\}$ is a filtration where each $\mathcal{F}_t$ contains all of the P -null sets in $\mathcal{F}$. Let $X \in L^1(\Omega, \mathcal{F}, P)$ and define $X_t = E[X|\mathcal{F}_{t+}]$ for all $t \geq 0$. Prove that $\{X_t, \mathcal{F}_{t+}, t \geq 0\}$ is a martingale. Prove that there is a modification Y of X such that $\{Y_t, \mathcal{F}_{t+}, t \geq 0\}$ is a right continuous martingale. (Most of the ingredients for the latter were proved in class, you just have to put them together in the right way.)

6*. Suppose that $\{X_t, \mathcal{F}_t, t \geq 0\}$ is a submartingale with right continuous paths. Prove that for all $\lambda > 0$ and $t \geq 0$,

$$\lambda P\left(\sup_{0 \leq s \leq t} X_s > \lambda\right) \leq E[X_t^+],$$

$$\lambda P\left(\inf_{0 \leq s \leq t} X_s < -\lambda\right) \leq E[X_t^+] - E[X_0].$$

Hint: For the first inequality for example, for $t \geq 0$ fixed, let $D_n = \{kt/2^n, k = 0, 1, 2, \dots, 2^n\}$. Note that by the right continuity of X and

definition of D_n , $\{\sup_{0 \leq s \leq t} X_s > \lambda\} \subset \cup_n \{\max_{s \in D_n} X_s > \lambda\}$. Apply a known result for discrete time submartingales.