

MATH 280C**HOMEWORK #1, DUE APRIL 8, 2015.**

Please hand in the starred problems. In the following, all random variables are defined on the probability space $(\Omega, \mathcal{F}, P)$.

1*. Let $\{\xi_i, i = 1, 2, \dots\}$ be a sequence of i.i.d. random variables with $\xi_i \in \{-1, 0, 1\}$ and $P(\xi_i = +1) = P(\xi_i = -1) = P(\xi_i = 0) = 1/3$. Let $S_0 = 0$, $S_n = \sum_{i=1}^n \xi_i$ for $n = 1, 2, \dots$ and let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma\{\xi_1, \dots, \xi_n\}$ for $n = 1, 2, \dots$. Fix integers $-a, b$ such that $-\infty < -a < 0 < b < \infty$. Let $T = \inf\{n \geq 0 : S_n = -a \text{ or } S_n = b\}$.

(a) Prove that T is a stopping time.

(b) Prove that $\{S_n, \mathcal{F}_n, n = 0, 1, 2, \dots\}$ is a martingale and that $\{S_n^2 - 2n/3, \mathcal{F}_n, n = 0, 1, 2, \dots\}$ is a martingale.

(c) Is the first martingale in (b) uniformly integrable? Give a reason to support your answer.

(d) Let $Y_n = S_{n \wedge T}$ and $Z_n = S_{n \wedge T}^2 - 2(n \wedge T)/3$ for $n = 0, 1, 2, \dots$. By Doob's simple stopping theorem, $\{Y_n, \mathcal{F}_n, n = 0, 1, 2, \dots\}$ and $\{Z_n, \mathcal{F}_n, n = 0, 1, 2, \dots\}$ are martingales. Show that

$$E[S_{n \wedge T}] = 0$$

and

$$E[S_{n \wedge T}^2] = 2E[n \wedge T]/3$$

for all n . (Be careful to show that you don't have ∞ on both sides in the last expression.)

(e) By taking suitable limits in the results of (d), and being careful to justify your steps, prove that $T < \infty$ a.s. and find expressions for $P(S_T = -a)$ and $P(S_T = b)$. Then obtain an expression for $E[T]$.

(f) By taking a limit as $-a$ tends to $-\infty$ in the results of (e), find $P(T_b < \infty)$ and $E[T_b]$ where $T_b = \inf\{n \geq 0 : S_n = b\}$. (You may want to relabel the original T as $T_{-a,b}$ and use the fact that $T_{-a,b} \uparrow T_b$ as $-a \rightarrow -\infty$.)

(g) Is the second martingale in (b) uniformly integrable? Give a reason to support your answer.

2*. Suppose that $\{\mathcal{F}_t, t \geq 0\}$ is a filtration. For each $t \geq 0$, let $\mathcal{G}_t = \mathcal{F}_{t+} = \cap_{s>t} \mathcal{F}_s$. Prove that $\{\mathcal{G}_t, t \geq 0\}$ is a right continuous filtration.

3*. Suppose that $(\Omega, \mathcal{F}, P)$ is a complete probability space and suppose that $\mathcal{G} \subset \mathcal{F}$ is a σ -algebra that contains all of the P -null sets in $\mathcal{F}$. Suppose that X is an $\mathcal{F}$ -measurable random variable and Y is a $\mathcal{G}$ -measurable random variable such that $X = Y$ a.s. (under P). Prove that X is also $\mathcal{G}$ -measurable.