

DUE MAY 20, 2015.

(1) Consider the canonical setup for Brownian motion, as described in class.

For each $a \geq 0$, let $\tau_a = \inf\{t \geq 0: X(t) - X(0) \geq a\}$.

From reflection principle results we have that for $a > 0$,

$$P_0(\tau_a \leq t) = 2 P_0(X(t) \geq a), \quad t \geq 0$$

(a) Prove that τ_a under P_0 and $a^2/2Z^2$ have the same distribution where

Z is a standard $N(0,1)$ random variable and $a > 0$ is fixed.

(b) Show that the distribution of τ_a under P_0 has a density and give a formula for this for $a > 0$ fixed.

(c) Show that $E_0[\tau_a] = +\infty$, $a > 0$.

(d) A common misuse of Doob's stopping theorem leads to the erroneous conclusion that $E_0[X_{\tau_a}] = E_0[X_0] = 0$ for $a > 0$.

Explain why you cannot use Doob's stopping theorem to obtain this relation.

(e) Prove the following identity for $0 \leq a \leq b < \infty$:

$$\tau_b = \tau_a + \tau_{b-a} \circ \theta_{\tau_a}$$

Use this and the strong Markov property of Brownian motion to prove that a.s. $\{\tau_a, a \geq 0\}$ under P_0 is a (non-decreasing) process that has stationary independent increments. If you are ambitious, show

that the paths of this process are a.s. right continuous with finite left limits. Prove that under P_0 , τ_a for $a > 0$ has

the same distribution as $a^2 \tau_1$. Use this to

prove that $\{\tilde{\tau}_a = b^2 \tau_{ba}, a \geq 0\}$ under P_0

has the same law as $\{\tau_a, a \geq 0\}$ under P_0 for every $b > 0$.

(Additional note:

The process $\{\tau_a, a \geq 0\}$ under P_0 is a non-decreasing stable process with parameter $1/2$. It is a.s. equal to a Lévy process with Lévy measure $\mu(dx) = dx/\sqrt{2\pi x^3}; x > 0$)

* (2) Let M_t and X_t under P_0 be as in the statement of the reflection principle:

$$P_0(M_t \geq a, X_t \leq b) = P_0(X_t \geq 2a - b)$$

for $t \geq 0$, $0 < a \leq b \leq \infty$.

Use this result to prove that under P_0 , for fixed $t \geq 0$,

$$M_t \triangleq \max_{0 \leq s \leq t} X(s), \quad |X_t| \quad \text{and} \quad M_t - X_t$$

all have the same distribution.

[Note: In fact, as stochastic processes, $\{M_t - X_t, t \geq 0\}$ and $\{|X_t|, t \geq 0\}$ are equivalent in law; however, these processes are not equivalent in law to $\{M_t, t \geq 0\}$. Note the latter is non-decreasing. The process $\{M_t - X_t, t \geq 0\}$ is used as a construction of $-X$ "reflected" at the origin.]

(3) Let B be a standard one-dimensional Brownian motion. Fix $a, b > 0$ positive constants.

Let

$$Z_t(t) = Z_0 e^{-at} + b e^{-at} B(e^{2at} - 1), \quad t \geq 0,$$

where Z_0 is a random variable independent of $B(\cdot)$.

[Z_t is an Ornstein-Uhlenbeck process.]

* (a) Prove that Z_t is a Gaussian process when Z_0 is Gaussian or $Z_0 = x$ and find the mean and covariance functions for Z_t .

* (b) Prove that as $t \rightarrow \infty$, $Z_t(t)$ converges in distribution to a Gaussian random variable with mean 0 and variance b^2 .

* (c) Prove that if Z_0 is a Gaussian rv with mean 0 + variance b^2 , then $Z_t(t)$ has the same distribution as $Z(0)$ for all $t \geq 0$.

3(d) For each $x \in \mathbb{R}$, let P_x denote the probability measure induced on $(\Omega = C([0, \infty), \mathbb{R}), \mathcal{M} = \sigma(\omega(s) : 0 \leq s < \infty))$ by Z_t when $Z_0 = x$, so that

$$P_x(A) = P(Z_t^x \in A) \text{ for each } A \in \mathcal{M} \text{ where } Z_t^x \text{ is the process with } Z_0^x = x.$$

Prove that the canonical process X on $(\Omega, \mathcal{M})$ with the family of probability measures $\{P_x, x \in \mathbb{R}\}$ satisfies the Markov property:

$$E_x^{P_x}[Y \circ \theta_t | \mathcal{F}_t] = E_x^{P_x}[Y] \quad P_x\text{-a.s.}$$

for all $t \geq 0$, where $\mathcal{F}_t = \mathcal{M}_{t+}$ and Y is any bounded random variable on $(\Omega, \mathcal{M})$.

3(e) Does the Feller continuity property hold for X under $\{P_x, x \in \mathbb{R}\}$; i.e., is the mapping

$$x \rightarrow E_x^{P_x}[f(X_t)]$$

continuous on $\mathbb{R}$ for each $t \geq 0$ and continuous, bounded function $f: \mathbb{R} \rightarrow \mathbb{R}$?

3(f) State and prove the strong Markov property for X under $\{P_x, x \in \mathbb{R}\}$.