

HOMEWORK 6 Due June 3, 2015.

*1. Let $\Omega = C([0, \infty), \mathbb{R}^2) = \{\omega: [0, \infty) \rightarrow \mathbb{R}^2\}$,

$$\mathcal{M} = \sigma\{\omega(s): 0 \leq s < \infty\}, \quad \mathcal{M}_t = \sigma\{\omega(s): 0 \leq s \leq t\}, \quad t \geq 0,$$

and let P_x denote the probability measure induced on $(\Omega, \mathcal{M})$ by two-dimensional Brownian motion starting from $x \in \mathbb{R}^2$.

Let $X(t)(\omega) = \omega(t)$, $\omega \in \Omega$, $t \geq 0$, + $\mathcal{F}_t = \mathcal{M}_t$ for all $t \geq 0$.

In a similar manner to that in problem 1 of HW #5, let

$$\tau_s = \inf\{t \geq 0: X_1(t) - X_1(0) \geq s\}, \quad s \geq 0.$$

Let $Y_s = X_2(\tau_s)$ for all $s \geq 0$

(i.e., $Y_s(\omega) = X_2(\tau_s(\omega))(\omega)$ for all $\omega \in \Omega$, $s \geq 0$).

(a) Show that the increments of Y have the stationary, independence property. (Hint: use the strong Markov property of X).

(b) Show, using the results of HW #5, problem 1, that the paths of Y are a.s. càdlàg.^(†)

(c) Show that Y is a.s. a Cauchy process (after (a), (b), you just need to show that the characteristic function for Y_s is $\lambda \mapsto \exp(-|\lambda|s)$, for all $s \geq 0$. For this, show that $\{\exp(i\lambda X_2(t) - |\lambda| X_1(t)), \mathcal{F}_t, t \geq 0\}$ is a martingale and use a suitable stopping theorem.

(†) càdlàg means right continuous with finite left limits.

2.* Let μ be a measure on $\mathbb{R} = (-\infty, \infty)$ with the Borel σ -algebra such that $\mu(\{0\}) = 0$ and $\int_{\mathbb{R}} (|x| \wedge 1) \mu(dx) < \infty$.

Let $\nu = \lambda \times \mu$ where λ is Lebesgue measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$

(a) Prove that $\mu((\varepsilon, \infty) \cup (-\infty, -\varepsilon)) < \infty$ for each $\varepsilon > 0$.

(b) Prove that ν is σ -finite on $(\mathbb{R} \times \mathbb{R}, \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{R}))$.
Give an example to show that ν need not be finite on $([0, \varepsilon] \times \mathbb{R}, \mathcal{B}([0, \varepsilon]) \times \mathcal{B}(\mathbb{R}))$.

(c) Let M be a Poisson random measure with mean measure ν .
Let $\tilde{Z}_t = \int_{[0, t] \times \mathbb{R}} |x| M(ds dx)$, $t \geq 0$.

Prove that $P(\tilde{Z}_t < \infty) = 1$ for each $t \geq 0$.

Hint: $P(\tilde{Z}_t < \infty) = \lim_{\lambda \downarrow 0} \mathbb{E}[e^{-\lambda \tilde{Z}_t}]$

$$\text{and } \tilde{Z}_t = \lim_{n \rightarrow \infty} \int_{[0, t] \times [\frac{1}{n}, \infty)} |x| M(ds dx) \\ \leq \sum_{k=1}^n \int_{[0, t] \times [\frac{1}{k}, \infty)} |x| M(ds dx)$$

where you can compute the Laplace transform of $\tilde{Z}_t^n$ for each n (note: $\mu^n = \mu|_{[\frac{1}{n}, \infty) \cup (-\infty, -\frac{1}{n}]}$ is a finite measure)

Let $Z_t = \int_{[0, t] \times \mathbb{R}} x M(ds dx)$ for $t \geq 0$.

(d) Prove that $\{Z_t, t \geq 0\}$ is a well defined process with stationary, independent increments and characteristic function

$$\mathbb{E}[e^{iKZ_t}] = e^{t \int_{\mathbb{R}} (e^{iKx} - 1) \mu(dx)} \quad K \in \mathbb{R}$$

(It can be shown that Z has a.s. a càdlàg modification.)

(e) (optional) Use a representation for $\tilde{Z}^m(t) = \int_{[0, t] \times (\frac{1}{m}, \infty) \cup (-\infty, -\frac{1}{m})} x M(ds dx)$ to show that Z moves only by jumps and is a.s. of bounded variation on each bounded time interval.