

MATH 280C**HOMEWORK #4, DUE May 6, 2015.**

Please hand in the starred problems. In the following, all random variables are defined on the probability space $(\Omega, \mathcal{F}, P)$.

1*. Fractional Brownian motion with a parameter $H \in (0, 1)$ is a Gaussian process $\{X(t), t \geq 0\}$ that starts from zero, has expectation zero for all time and has the following covariance function:

$$R(s, t) = E[X(t)X(s)] = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}),$$

for all $s, t \geq 0$. Prove that there is a continuous modification of X . (Extra credit: In fact, there is a modification of X whose paths are Hölder continuous with exponent $H - \epsilon$ for all $\epsilon \in (0, H)$.)

Remark: For $H > 1/2$, fractional Brownian motion exhibits long range dependence, i.e.,

$$\sum_{n=1}^{\infty} E[X(1)(X(n+1) - X(n))] = +\infty.$$

2*. Suppose that B is a standard one-dimensional Brownian motion.

(a) Prove that a.s.,

$$\limsup_{t \rightarrow \infty} \frac{B_t}{\sqrt{t}} = +\infty$$

and

$$\liminf_{t \rightarrow \infty} \frac{B_t}{\sqrt{t}} = -\infty$$

Hint: Consider positive integer valued t first and use Kolmogorov's zero-one law.

(b) Use a transformation of Brownian motion using time inversion to prove that

$$\limsup_{t \rightarrow 0} \frac{B_t}{\sqrt{t}} = +\infty$$

and

$$\liminf_{t \rightarrow 0} \frac{B_t}{\sqrt{t}} = -\infty.$$

Use this to argue that a.s. the paths of B are not Hölder continuous of order one-half at $t = 0$.