

MATH 280C

HOMEWORK #3, DUE April 29, 2015.

Please hand in the starred problems. In the following, all random variables are defined on the probability space $(\Omega, \mathcal{F}, P)$.

1*. Suppose that $\{M_t, \mathcal{F}_t, t \geq 0\}$ is a right continuous uniformly integrable martingale and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, t \geq 0)$ satisfies the usual conditions. Prove that M is of class D , i.e., $\{M_T : T \text{ is a real-valued stopping time}\}$ is uniformly integrable.

2*. A standard one-dimensional Brownian motion is a real-valued stochastic process $\{B_t, t \geq 0\}$ such that

- $B(0) = 0$,
- for any positive integer n and times $0 \leq t_1 < t_2 < \dots < t_n$, $\{B(t_1), B(t_2) - B(t_1), \dots, B(t_n) - B(t_{n-1})\}$ are independent,
- for each $0 \leq s < t < \infty$, $B(t) - B(s)$ has a normal distribution with mean zero and variance $t - s$,
- the paths of B are continuous.

(This is a process with stationary, independent increments).

For each $t \geq 0$, let $\mathcal{H}_t = \sigma\{B(s) : 0 \leq s \leq t\}$, the smallest σ -algebra with respect to which $B(s)$ is measurable for each $0 \leq s \leq t$. Suppose that $(\Omega, \mathcal{F}, P)$ is complete and let $\mathcal{N}$ denote the P -null sets in $\mathcal{F}$. Let $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{N}$, the smallest σ -algebra containing $\mathcal{H}_t$ and $\mathcal{N}$. Let $\mathcal{F}_t = \mathcal{G}_{t+}$ for all $t \geq 0$. In class we showed that $\{B(t), \mathcal{H}_t, t \geq 0\}$ and $\{B_t^2 - t, \mathcal{H}_t, t \geq 0\}$ are martingales.

(a) Prove that $\{B(t), \mathcal{F}_t, t \geq 0\}$ and $\{B_t^2 - t, \mathcal{F}_t, t \geq 0\}$ are martingales. Hint: first prove this with $\mathcal{G}_t$ in place of $\mathcal{F}_t$.

(b) Let $T = \inf\{t \geq 0 : B_t > 1\}$. Is T a stopping time relative to $\{\mathcal{F}_t, t \geq 0\}$? Make sure to justify your answer.

(c) Fix $a < 0 < b$ and let $S = \inf\{s \geq 0 : B_s \leq a \text{ or } B_s \geq b\}$. Is S a stopping time relative to $\{\mathcal{F}_t, t \geq 0\}$? Use Doob's stopping theorem to show that $E[S] < \infty$, $E[B_S] = 0$ and to compute the probability that B hits a before b . You should also be able to compute $E[S]$.

(d) (Extra credit) Prove that $\{M_t = \exp(cB_t - \frac{1}{2}c^2t), \mathcal{F}_t, t \geq 0\}$ is a martingale for any real number c . (You may use results about the moment generating function for the normal distribution.) Prove that $\lim_{t \rightarrow \infty} M_t$ exists a.s. Can you identify this limit?

3*. Let B be a standard one-dimensional Brownian motion. Let $X(t) = B(t) - tB(1)$ for all $0 \leq t \leq 1$. The process X is called a *Brownian bridge*.

- (a) Show that $\{X(t) : 0 \leq t \leq 1\}$ is a Gaussian process and compute its covariance function $R(s, t) = \text{Cov}(X(s), X(t))$ for $0 \leq s, t \leq 1$.
- (b) (Extra credit). Show that for $0 < t_1 < t_2 < \dots < t_n < 1$, the joint distribution of $(B(t_1), B(t_2), \dots, B(t_n))$ given $|B(1)| \leq \epsilon$, converges to the joint distribution of $(X(t_1), \dots, X(t_n))$, as $\epsilon \downarrow 0$.