

HOMEWORK #3

DUE

WED. OCT. 29 IN CLASS.

1. Let $(\Omega, \mathcal{F})$ be a measurable space.

Prove that $A \in \mathcal{F}$ if and only if $1_A: \Omega \rightarrow \mathbb{R}$ is measurable where $\mathbb{R}$ has the Borel σ -algebra on it.

$$\text{Here } 1_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{if } \omega \notin A. \end{cases}$$

*2. Suppose $\Omega = [0, 1]$, $\mathcal{F} = \mathcal{B}([0, 1])$, $P = \text{Lebesgue measure on } (\Omega, \mathcal{F})$.

Define

$$X_1(\omega) = 0 \quad \text{for all } \omega \in \Omega,$$

$$X_2(\omega) = 1_{\{1/3\}}(\omega) \quad \text{for all } \omega \in \Omega,$$

$$X_3(\omega) = 1_A(\omega) \quad \text{for all } \omega \in \Omega$$

where A is the set of rational numbers in $[0, 1]$.

(i) Find $\sigma(X_i)$ for $i=1, 2, 3$, the σ -algebras on Ω generated by X_1, X_2, X_3 .

(ii) Prove that

$$P(X_1 = X_2 = X_3) = 1.$$

(Hint: You may use the fact that the Lebesgue measure of the rationals is zero).

*3. Prove that if X is a random variable, so is $|X|$.
Give an example to show that the converse may be false.

*4. Exercise 1.3.5 from Durrett.

5. Suppose $X: \Omega \rightarrow \mathbb{R}$ takes only countably many values in $\mathbb{R}$. Show that X is measurable if and only if

$$X^{-1}(\{x\}) \in \mathcal{F} \text{ for each } x \in \mathbb{R}.$$

[Here Ω has σ -algebra $\mathcal{F}$ and $\mathbb{R}$ has the Borel σ -algebra $\mathcal{B}(\mathbb{R})$.]

6. Let X and Y be random variables and let $A \in \mathcal{F}$.
Prove that

$$Z(\omega) = \begin{cases} X(\omega) & \text{if } \omega \in A, \\ Y(\omega) & \text{if } \omega \in A^c, \end{cases}$$

is a random variable.

[Here X, Y, Z are all defined on the measurable space $(\Omega, \mathcal{F})$.]