

HOMEWORK 2: Please hand in starred problems by Wed, Oct. 22, in class.

\* Exercise 1:

Let  $\Omega$  be a non-empty set.

Suppose  $\mathcal{L}$  is a non-empty collection of subsets of  $\Omega$ .

Suppose  $\mathcal{L}$  satisfies

(i)  $\Omega \in \mathcal{L}$ ;

(ii) if  $A \in \mathcal{L}$  then  $A^c \in \mathcal{L}$ ;

(iii) if  $\{A_n\}_{n=1}^{\infty}$  is a collection of disjoint sets in  $\mathcal{L}$  then  $\bigcup_{n=1}^{\infty} A_n$  is in  $\mathcal{L}$ .

Prove that  $\mathcal{L}$  is a  $\lambda$ -system.

[Comment: in fact the above is an equivalent characterization to the one given in class.]

Exercise 2:

Let  $\Omega = \{1, 2, 3, 4\}$ ,  $\mathcal{C} = \{\{1, 2, 3\}, \{2, 3\}\}$ .

Find  $\sigma(\mathcal{C})$ .

Exercise 3: Let  $\Omega$  be a non-empty set.

A monotone class  $\mathcal{M}$  is a non-empty collection of subsets of subsets of  $\Omega$  that is closed under monotone limits, i.e., (i) if  $\{A_n\}_{n=1}^{\infty}$  is in  $\mathcal{M}$  with  $A_n \subset A_{n+1}$  for all  $n$ , then  $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$  and (ii) if  $\{A_n\}_{n=1}^{\infty}$  is in  $\mathcal{M}$  and  $A_n \supset A_{n+1}$  for all  $n$ , then  $\bigcap_{n=1}^{\infty} A_n \in \mathcal{M}$ .

Show that a  $\sigma$ -algebra is an algebra that is also a monotone class, and conversely, an algebra that is a monotone class is a  $\sigma$ -algebra.

Exercise 4: Suppose  $\mathcal{P}$  is a  $\Pi$ -system contained in a monotone class  $\mathcal{M}$ . Give an example to show that  $\sigma(\mathcal{P})$  need not be contained in  $\mathcal{M}$ . Hint: Consider  $\Omega$  with a small number of elements.

②

\* Exercise 5. Suppose  $\mathcal{F}$  is a  $\sigma$ -algebra of subsets of a non-empty set  $\Omega$ . Suppose  $Q: \mathcal{F} \rightarrow [0, 1]$  is a function such that

- (i)  $Q$  is finitely additive on  $\mathcal{F}$ ;
- (ii)  $0 \leq Q(A) \leq 1$  for all  $A \in \mathcal{F}$  and  $Q(\Omega) = 1$ ;
- (iii) if  $\{A_i\}_{i=1}^{\infty}$  are disjoint sets in  $\mathcal{F}$  and  $\Omega = \biguplus_{i=1}^{\infty} A_i$ , then  $\sum_{i=1}^{\infty} Q(A_i) = 1$ .

Prove that  $Q$  is a probability measure (i.e. show  $Q$  is countably additive on  $\mathcal{F}$ ).

\* Exercise 6. Let  $\Omega = \{1, 2, 3, \dots\}$ .

Suppose  $\mathcal{A}$  is the algebra of subsets of  $\Omega$  containing all sets  $A$  such that  $A$  or  $A^c$  is finite.

Let  $\mu: \mathcal{A} \rightarrow [0, 1]$  be defined by

$$\mu(A) = \begin{cases} 0 & \text{if } A \text{ is finite} \\ 1 & \text{if } A^c \text{ is finite.} \end{cases}$$

(i) Prove that  $\mu$  is finitely additive on  $\mathcal{A}$ .

(ii) Give an example of a sequence of sets  $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$  such that  $A_n \downarrow \emptyset$  as  $n \rightarrow \infty$  (i.e.,  $A_n \supset A_{n+1}$  for all  $n$  and  $\bigcap_n A_n = \emptyset$ ) and  $\mu(A_n) \not\rightarrow 0$  as  $n \rightarrow \infty$ .