

Homework #4 due Wednesday, Nov. 5 in class.

$(\Omega, \mathcal{F}, P)$ is a probability space.

*1. Suppose $X_1, \dots, X_n$ are (real-valued) random variables defined on Ω . Prove that

(a) $\max(X_1, \dots, X_n)$ is a random variable.

(b) $\min(X_1, \dots, X_n)$ is a random variable.

For (b) try to use the result from (a).

2. Durrett: Ex. 1.3.1

3. Durrett: Ex. 1.3.3

*4. Suppose $X: \Omega \rightarrow \mathbb{R}$ is a function.

Prove that $Y: \Omega \rightarrow \mathbb{R}$ is $\sigma(X)/\mathcal{B}(\mathbb{R})$ -measurable if and only if $Y = f(X)$ for some Borel measurable function $f: \mathbb{R} \rightarrow \mathbb{R}$.

Hint: Use a monotone class theorem for random variables. Take care to show that f can be chosen to be real valued.

5. Durrett, Ex. 2.3.10

*6. Durrett, Ex. 2.3.11. In the last part of this problem, the example is to be such that Borel Cantelli cannot be directly applied to $\{A_n\}$.

7. (see next page)

7. Suppose that $F: \mathbb{R} \rightarrow [0, 1]$ is a cumulative distribution function. Define $G: (0, 1) \rightarrow \mathbb{R}$ by

$$G(y) = \inf \{x: F(x) \geq y\} \quad \text{for } 0 < y < 1.$$

(a) Prove that $G(y) > x$ if and only if $y > F(x)$ for all $y \in (0, 1)$, $x \in \mathbb{R}$.

(You may find it helpful to draw graphs of distribution functions.)

(b) Suppose that $Y: \Omega \rightarrow (0, 1)$ is a uniformly distributed random variable on $(0, 1)$.

Let $X = G(Y): \Omega \rightarrow \mathbb{R}$.

Prove that X has distribution function F .

Hint: Prove that $P(X > x) = 1 - F(x)$ for all $x \in \mathbb{R}$, using (a).

Remarks:

G is an "inverse" for F (F is not necessarily invertible, so this is a "kind of inverse"). This is in fact the same as the left continuous inverse function F^{-1} introduced in Theorem 1.2.2 of Durrett. (This can be proved using (a) above). Part (b) provides a mechanism for simulating a random variable with distribution function F .

(c) Suppose Z is a random variable with distribution function F and F is continuous. Use (a) to prove that $W = F(Z)$ is a uniform random variable on $(0, 1)$.

Hint: Prove that $F(G(y)) = y$ for all $y \in (0, 1)$, when F is continuous.