

Multi-dimensional Black-Scholes Model

In this chapter, we focus on a continuous financial market model that generalizes the Black-Scholes model of Chapter 4 in several directions. The model considered here has finitely many assets. One of these is a “riskless” money market asset and the others can be viewed as “risky” stocks. The price processes for the assets are given by an Itô process (cf. Appendix D) where the money market asset price process is non-decreasing with a zero volatility coefficient. Apart from this constraint, the drift and volatility coefficients of the Itô process can be stochastic and time dependent.

Following usage by some other authors [32], we call the (primary) financial market model considered here a multi-dimensional Black-Scholes model. This may be viewed as a continuous analogue of the finite market model studied in Chapter 3. One can consider more general financial market models using continuous semimartingales, or even general semimartingales with jumps, for the price processes. However, consideration of more general models would take us beyond the introductory level of this book. Moreover, the multi-dimensional Black-Scholes model considered here does allow us to illustrate how the fundamental theorems of asset pricing developed for the finite market model in Chapter 3 can be extended to a class of continuous models.

We begin by defining the multi-dimensional Black-Scholes model and the notion of a self-financing trading strategy. There are several different ways of defining such a model depending on the degree of measurability and integrability that one assumes for the coefficients in the Itô process and for the trading strategies. Our integrability assumptions on the trading

strategies are similar to those used in Karatzas and Shreve [28], Section 1.2. This form will be useful when proving some results related to completeness of the model.

After defining the financial market model, we describe the first fundamental theorem of asset pricing. Extrapolating from the finite market model case, we would expect this to say essentially that absence of arbitrage opportunities in the market model is equivalent to the existence of a risk neutral probability. The precise formulation of a suitably weak notion of arbitrage and of a risk neutral probability (as an equivalent *local* martingale measure) are subtle in the continuous setting. Although versions of the first fundamental theorem of asset pricing go back to the seminal works of Harrison, Kreps and Pliska [20, 21, 22, 29], a full understanding of this equivalence in a very general setting for price processes was developed only in the 1990s. Here we state a form of the fundamental theorem of asset pricing that derives from a 1994 result of Delbaen and Schachermayer [14]. Following that, we characterize the form of equivalent local martingale measures for the multi-dimensional Black-Scholes model. This exploits the Itô process form of the asset price processes and the (local) martingale representation theorem for Brownian motion.

Assuming that the market model is viable, which by the first fundamental theorem amounts to assuming that there exists an equivalent local martingale measure, we then describe the second fundamental theorem of asset pricing. This asserts that all suitably integrable European contingent claims are replicable (i.e., the market is complete) if and only if there is a *unique* equivalent local martingale measure. Assuming a certain non-degeneracy (i.e., there are no more stocks than there are degrees of freedom in the underlying Brownian motion driving the asset price processes), we show that the market is complete if and only if there are exactly as many stocks as there are degrees of freedom in the driving Brownian motion and almost surely the volatility matrix for the stocks is invertible at almost every time. There are some subtleties associated with these results on completeness that relate to integrability assumptions on self-financing strategies (see Harrison and Pliska [21], Section 3.3, for one indication of this). Our use of a similar setup to that of Karatzas and Shreve [28] is convenient for handling these subtleties. For more general market models (not treated here), completeness need not imply uniqueness of an equivalent local martingale measure (cf. Artzner and Heath [1] for an example), and it becomes a challenge to formulate a suitable variant of the second fundamental theorem (cf. Jarrow, Jin and Madan [26] for an approach to this and for a description of some of the related history).

Assuming the market model is viable and complete, we illustrate how the unique arbitrage free price for a suitably integrable European contingent claim can be obtained. We conclude the chapter with some references to the literature on incomplete markets.

5.1. Preliminaries

We use a similar basic setup to that of the Black-Scholes model treated in Chapter 4, except that now we have a multi-dimensional Brownian motion as a driving noise source.

We consider a finite time interval $[0, T]$, for some $0 < T < \infty$, as the interval during which trading may take place. The Borel σ -algebra of subsets of $[0, T]$ will be denoted by $\mathcal{B}_T$.

We assume as given a complete probability space $(\Omega, \mathcal{F}, P)$ on which is defined a standard n -dimensional *Brownian motion* $W = \{W_t, t \in [0, T]\}$ for some positive integer n . In particular, $W = (W^1, \dots, W^n)$ is an n -dimensional process defined on the time interval $[0, T]$; the one-dimensional coordinate processes $W^1, \dots, W^n$ are mutually independent; and for each $i = 1, \dots, n$, $W^i = \{W_t^i, t \in [0, T]\}$ is a standard one-dimensional Brownian motion. Let $\{\mathcal{F}_t, t \in [0, T]\}$ denote the standard filtration generated by the n -dimensional Brownian motion W under P (cf. Appendix D, Section D.1). It is well known that this filtration is right continuous; i.e., for each $t \in [0, T)$, $\mathcal{F}_t = \mathcal{F}_{t+} \equiv \bigcap_{s \in (t, T]} \mathcal{F}_s$ (cf. Chung [9], Section 2.3, Theorem 4). The σ -algebra $\mathcal{F}_0$ is trivial in the sense that any $\mathcal{F}_0$ -measurable random variable is P -a.s. constant. All random variables considered in this chapter will be assumed to be defined on $(\Omega, \mathcal{F}_T)$, and so without loss of generality we assume that $\mathcal{F} = \mathcal{F}_T$. We shall frequently write $\{\mathcal{F}_t\}$ instead of the more cumbersome $\{\mathcal{F}_t, t \in [0, T]\}$. The filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ satisfies the usual conditions (cf. Appendix C). It is well known that under this assumption, every martingale has a modification whose paths are all right continuous with finite left limits (cf. Chung [9], Theorem 3, page 29, and Corollary 1, page 26). In fact, since the filtration is generated by the Brownian motion W , every (local) martingale has a continuous modification (cf. Theorems II.2.9, V.3.5 in Revuz and Yor [35]). Expectations with respect to P will be denoted by $E[\cdot]$ unless there is more than one probability measure under consideration, in which case we shall use $E^P[\cdot]$ in place of $E[\cdot]$.

In the following, we shall encounter integrals of the form $\int_0^t R_s ds$, for $t \in [0, T]$, where $R = \{R_t, t \in [0, T]\}$ is an m -dimensional, adapted process (for some $m \geq 1$) satisfying

- (i) $R : [0, T] \times \Omega \rightarrow \mathbb{R}^m$ is $(\mathcal{B}_T \times \mathcal{F}_T)$ -measurable, where $R(t, \omega) = R_t(\omega)$ for each $t \in [0, T]$ and $\omega \in \Omega$;
- (ii) $\int_0^T |R_s| ds < \infty$ P -a.s., where $|x| = (\sum_{i=1}^m (x^i)^2)^{\frac{1}{2}}$ for each $x = (x^1, \dots, x^m) \in \mathbb{R}^m$.

Given these assumptions, P -a.s., the integral $\int_0^t R_s ds$ is well defined pathwise (and componentwise) for all $t \in [0, T]$. For convenience, on an exceptional P -null set, we define these integrals for all $t \in [0, T]$ to be zero. One can show that this defines a continuous adapted process $\{\int_0^t R_s ds, t \in [0, T]\}$ (cf. Appendix D, Section D.3). Whenever we encounter such pathwise integrals, we shall always assume that they have been defined suitably on an exceptional null set.

5.2. Multi-dimensional Black-Scholes Model

Our multi-dimensional Black-Scholes model has $d + 1$ assets where d is a strictly positive integer. One of the assets is viewed as a money market instrument that has a possibly stochastic and time-dependent rate of return. The other d assets are viewed as stocks.

The *money market* price process, denoted by $B = \{B_t, t \in [0, T]\}$, is a continuous process given by

$$B_t = \exp \left(\int_0^t r_s ds \right), \quad t \in [0, T], \quad (5.1)$$

where $r = \{r_t, t \in [0, T]\}$ is an adapted process taking values in $[0, \infty)$ such that

- (i) $r : [0, T] \times \Omega \rightarrow [0, \infty)$ is $(\mathcal{B}_T \times \mathcal{F}_T)$ -measurable where $r(t, \omega) = r_t(\omega)$ for each $t \in [0, T]$ and $\omega \in \Omega$, and
- (ii) $\int_0^T r_t dt < \infty$ P -a.s.

Remark. The convention mentioned at the end of the previous section is used in defining $\{\int_0^t r_s ds, t \in [0, T]\}$ as a continuous adapted process.

We may think of r_t as the short-term interest rate for the money market instrument at time t . The process $r = \{r_t, t \in [0, T]\}$ is sometimes called the *rate of return process*. The money market process satisfies $B_0 = 1$, and the following dynamic equation holds P -a.s.:

$$B_t = 1 + \int_0^t r_s B_s ds, \quad t \in [0, T]. \quad (5.2)$$

Since B has continuous paths that are non-decreasing (as $r_t \geq 0$ for all t), the money market asset is sometimes referred to as a riskless asset, although its rate of return process r can be stochastic (in a non-anticipating manner). We shall use the money market asset as a *numeraire*; i.e., a reference price

process which is used to normalize the other price processes in forming discounted values. In the Black-Scholes model of Chapter 4, the bond played the role of numeraire.

We assume that for some fixed positive integer d there are d *stocks* with continuous, adapted price processes $S^i = \{S_t^i, t \in [0, T]\}$, $i = 1, \dots, d$, that take values in $(0, \infty)$ and are such that for each i and $t \in [0, T]$,

$$S_t^i = S_0^i \exp \left(\int_0^t \left(\mu_s^i - \frac{1}{2} \sigma_s^i \cdot \sigma_s^i \right) ds + \int_0^t \sigma_s^i \cdot dW_s \right), \quad (5.3)$$

where S_0^i is a strictly positive constant and μ^i and σ^i are given such that μ^i is the i^{th} component of a d -dimensional process $\mu = \{\mu_t, t \in [0, T]\}$ and $\sigma^i = \{\sigma_t^i, t \in [0, T]\}$ is the i^{th} row of a $(d \times n)$ -matrix-valued process $\sigma = \{\sigma_t, t \in [0, T]\}$, and μ and σ are assumed to satisfy the following:

- (i) $\mu = \{\mu_t, t \in [0, T]\}$ is a d -dimensional, adapted process such that $\mu : [0, T] \times \Omega \rightarrow \mathbb{R}^d$ is $(\mathcal{B}_T \times \mathcal{F}_T)$ -measurable where $\mu(t, \omega) = \mu_t(\omega)$ for all $t \in [0, T]$ and $\omega \in \Omega$, and $\int_0^T |\mu_t| dt < \infty$ P -a.s., where $|\mu_t| = (\sum_{i=1}^d (\mu_t^i)^2)^{\frac{1}{2}}$ for $t \in [0, T]$;
- (ii) $\sigma = \{\sigma_t, t \in [0, T]\}$ is an adapted process taking values in the set of $d \times n$ matrices with real entries (denoted by $\mathbb{R}^{d \times n}$), $\sigma : [0, T] \times \Omega \rightarrow \mathbb{R}^{d \times n}$ is $(\mathcal{B}_T \times \mathcal{F}_T)$ -measurable where $\sigma(t, \omega) = \sigma_t(\omega)$ for all $t \in [0, T]$ and $\omega \in \Omega$, and $\int_0^T |\sigma_t|^2 dt < \infty$ P -a.s., where $|\sigma_t|^2 = \sum_{i=1}^d \sum_{j=1}^n |\sigma_t^{ij}|^2$ for $t \in [0, T]$ and σ^{ij} denotes the ij component of the $d \times n$ -matrix-valued process σ .

These assumptions ensure that the exponent in (5.3) defines the i^{th} component of an Itô process driven by the Brownian motion W (cf. Appendix D, Section D.3):

$$X_t^i = \int_0^t \left(\mu_s^i - \frac{1}{2} \sigma_s^i \cdot \sigma_s^i \right) ds + \int_0^t \sigma_s^i \cdot dW_s, \quad t \in [0, T],$$

for $i = 1, \dots, d$. Using Itô's formula (cf. Appendix D, Section D.4) on the Itô process with the function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ given by $f(x) = \exp(x^i)$ for $x \in \mathbb{R}^d$, we see that for $i = 1, \dots, d$, S^i satisfies the following dynamic equation P -a.s.:

$$S_t^i = S_0^i + \int_0^t S_s^i \mu_s^i ds + \int_0^t S_s^i \sigma_s^i \cdot dW_s, \quad t \in [0, T]. \quad (5.4)$$

Indeed, this shows that the d -dimensional stock price process $(S^1, \dots, S^d)$ is itself an Itô process driven by the Brownian motion W . Moreover, the process $(B, S^1, \dots, S^d)$ is a $(d+1)$ -dimensional Itô process, where the first component is a non-decreasing process.

For brevity, it is sometimes useful to write the equations (5.2) and (5.4), satisfied by the asset price processes, in differential form:

$$dB_t = r_t B_t dt, \quad t \in [0, T], \quad (5.5)$$

$$dS_t^i = S_t^i \mu_t^i dt + S_t^i \sigma_t^i \cdot dW_t, \quad t \in [0, T], \quad i = 1, \dots, d. \quad (5.6)$$

We note for future reference that the process $B^{-1} = \{B_t^{-1}, t \in [0, T]\}$ where

$$B_t^{-1} = \exp\left(-\int_0^t r_s ds\right), \quad t \in [0, T], \quad (5.7)$$

is also an Itô process which satisfies the following dynamic equation P -a.s. for all $t \in [0, T]$:

$$B_t^{-1} = 1 - \int_0^t r_s B_s^{-1} ds. \quad (5.8)$$

For convenience, we call the money market instrument asset 0 and define $S_t^0 = B_t$ for all $t \in [0, T]$. We shall henceforth use B and S^0 interchangeably and we let $S = (S^0, S^1, \dots, S^d)$. We define the *discounted asset price process* $S^* = (S^{*,0}, S^{*,1}, \dots, S^{*,d})$ by

$$S_t^{*,i} = \frac{S_t^i}{B_t} = S_t^i \exp\left(-\int_0^t r_s ds\right), \quad t \in [0, T], \quad i = 0, 1, \dots, d. \quad (5.9)$$

In particular, $S^{*,0} \equiv 1$. Using Itô's formula (cf. Appendix D) for the $(d+1)$ -dimensional Itô process (B^{-1}, S) with the function $f(x, y) = xy^i$ for $x \in \mathbb{R}$ and $y \in \mathbb{R}^d$, together with (5.4) and (5.8), we see that for $i = 1, \dots, d$,

$$dS_t^{*,i} = S_t^{*,i} (\mu_t^i - r_t) dt + S_t^{*,i} \sigma_t^i \cdot dW_t, \quad t \in [0, T], \quad (5.10)$$

or in other words, P -a.s., for all $t \in [0, T]$,

$$S_t^{*,i} = S_0^{*,i} + \int_0^t S_s^{*,i} (\mu_s^i - r_s) ds + \int_0^t S_s^{*,i} \sigma_s^i \cdot dW_s. \quad (5.11)$$

A *trading strategy* is a $(d+1)$ -dimensional process

$$\phi = \{\phi_t = (\phi_t^0, \phi_t^1, \dots, \phi_t^d), t \in [0, T]\}$$

such that

- (i) $\phi : [0, T] \times \Omega \rightarrow \mathbb{R}^{d+1}$ is $(\mathcal{B}_T \times \mathcal{F}_T)$ -measurable where $\phi(t, \omega) = \phi_t(\omega)$ for $t \in [0, T]$ and $\omega \in \Omega$;
- (ii) ϕ is adapted, i.e., $\phi_t \in \mathcal{F}_t$ for each $t \in [0, T]$;
- (iii)
 - (a) $\int_0^T |\phi_t \cdot S_t| r_t dt < \infty$ P -a.s.,
 - (b) $\int_0^T \left| \sum_{i=1}^d \phi_t^i S_t^i (\mu_t^i - r_t) \right| dt < \infty$ P -a.s.,
 - (c) $\int_0^T \left| \sum_{i=1}^d \phi_t^i S_t^i \sigma_t^i \right|^2 dt < \infty$ P -a.s.

These conditions ensure that the integrals used below in defining the notion of self-financing yield continuous adapted processes (after possibly defining them suitably on an exceptional P -null set). Following Karatzas and Shreve [28], we have used a minimal set of integrability conditions on ϕ to facilitate some later manipulations associated with establishing conditions for completeness of the market model (cf. Section 5.5).

Here ϕ_t^0 is interpreted as the number of shares of the money market instrument held at time t , and for $i = 1, \dots, d$, ϕ_t^i is interpreted as the number of shares of stock i held at time t . The *value* at time $t \in [0, T]$ of the portfolio associated with ϕ is given by

$$V_t(\phi) = \phi_t \cdot S_t = \sum_{i=0}^d \phi_t^i S_t^i. \quad (5.12)$$

Remark. The σ -algebra $\mathcal{F}_0$ is trivial; i.e., it contains only sets of probability zero or one. It follows that the $\mathcal{F}_0$ -measurable random variable $V_0(\phi)$ is constant P -a.s. Thus, without loss of generality, by redefining ϕ_0 on a P -null set if necessary, we may and do assume that $V_0(\phi)$ is identically constant.

We define the *discounted value process*

$$V_t^*(\phi) = \frac{V_t(\phi)}{B_t}, \quad t \in [0, T]. \quad (5.13)$$

Note that, since $B_0 = 1$, $V_0^*(\phi) = V_0(\phi)$.

A trading strategy ϕ is said to be *self-financing* if $V(\phi) = \{V_t(\phi), t \in [0, T]\}$ is a continuous, adapted process such that P -a.s., for all $t \in [0, T]$,

$$\begin{aligned} V_t(\phi) &= V_0(\phi) + \int_0^t \phi_s \cdot S_s r_s ds + \int_0^t \left(\sum_{i=1}^d \phi_s^i S_s^i (\mu_s^i - r_s) \right) ds \\ &\quad + \int_0^t \left(\sum_{i=1}^d \phi_s^i S_s^i \sigma_s^i \right) \cdot dW_s. \end{aligned} \quad (5.14)$$

As mentioned above, the measurability and integrability assumptions placed on a trading strategy ϕ ensure that the integrals in the expression above can be used to define continuous adapted processes. We note that (5.14) can be rewritten as follows, which shows that $V(\phi)$ is a one-dimensional Itô process:

$$\begin{aligned} V_t(\phi) &= V_0(\phi) + \int_0^t \left(\phi_s^0 S_s^0 r_s + \sum_{i=1}^d \phi_s^i S_s^i \mu_s^i \right) ds \\ &\quad + \int_0^t \left(\sum_{i=1}^d \phi_s^i S_s^i \sigma_s^i \right) \cdot dW_s. \end{aligned} \quad (5.15)$$

With a suitable splitting up of the integral with respect to S , one may formally interpret the self-financing condition as representing the condition:

$$dV_t(\phi) = \phi_t \cdot dS_t, \quad t \in [0, T].$$

In other words, the self-financing condition requires that changes in the value of the portfolio result only from changes in the values of the assets.

Similar to Lemma 4.2.1 in Chapter 4, we can characterize self-financing in terms of the discounted value process as follows.

Lemma 5.2.1. *A trading strategy ϕ is self-financing if and only if $V^*(\phi)$ is a continuous, adapted process such that P -a.s., for each $t \in [0, T]$,*

$$\begin{aligned} V_t^*(\phi) = & V_0(\phi) + \int_0^t \left(\sum_{i=1}^d \phi_s^i S_s^{*,i} (\mu_s^i - r_s) \right) ds \\ & + \int_0^t \left(\sum_{i=1}^d \phi_s^i S_s^{*,i} \sigma_s^i \right) \cdot dW_s. \end{aligned} \quad (5.16)$$

Remark. The integrals on the right side of equation (5.16) are well defined by the conditions on ϕ , since the additional multiplicative factor in the integrands provided by the positive continuous process B^{-1} is P -a.s. bounded away from zero and infinity.

One may formally interpret the condition (5.16) as requiring that

$$dV_t^*(\phi) = \phi_t \cdot dS_t^*, \quad t \in [0, T]. \quad (5.17)$$

Proof. Note that $V(\phi)$ is continuous and adapted if and only if $V^*(\phi)$ has this property. Thus it suffices to show that, assuming this property, (5.14) holds if and only if (5.16) holds. Just as with Lemma 4.2.1, the proof of this involves simple manipulations using stochastic calculus. We shall give the proof of the “only if” part; the proof of the “if part” is very similar.

Suppose that ϕ is a self-financing trading strategy. Then $V(\phi)$ is an Itô process that satisfies the dynamic equation (5.14). Thus, $(B^{-1}, V(\phi))$ is a two-dimensional Itô process; and applying Itô’s formula to this process with the function $f(x, y) = xy$ for $(x, y) \in \mathbb{R}^2$, we obtain P -a.s., for each

$t \in [0, T]$,

$$\begin{aligned}
V_t^*(\phi) &\equiv B_t^{-1} V_t(\phi) \\
&= V_0(\phi) + \int_0^t V_s(\phi) dB_s^{-1} + \int_0^t B_s^{-1} dV_s(\phi) \\
&= V_0(\phi) - \int_0^t V_s(\phi) B_s^{-1} r_s ds + \int_0^t B_s^{-1} \phi_s \cdot S_s r_s ds \\
&\quad + \int_0^t B_s^{-1} \left(\sum_{i=1}^d \phi_s^i S_s^i (\mu_s^i - r_s) \right) ds \\
&\quad + \int_0^t B_s^{-1} \left(\sum_{i=1}^d \phi_s^i S_s^i \sigma_s^i \right) \cdot dW_s \\
&= V_0(\phi) + \int_0^t \left(\sum_{i=1}^d \phi_s^i S_s^{*,i} (\mu_s^i - r_s) \right) ds \\
&\quad + \int_0^t \left(\sum_{i=1}^d \phi_s^i S_s^{*,i} \sigma_s^i \right) \cdot dW_s,
\end{aligned}$$

where the first two integrals in the third line cancel one another because $V_s(\phi) = \phi_s \cdot S_s$, and we have used the fact that $S_s^{*,i} = B_s^{-1} S_s^i$. Thus, $V^*(\phi)$ satisfies (5.16) and it too is an Itô process. $\square$

We note that the right side of equation (5.16) only involves the discounted stock price processes, $S^{*,1}, \dots, S^{*,d}$. Indeed, this observation enables us to specify a self-financing strategy simply by specifying the components corresponding to the stocks. More precisely, let $\hat{S}^* = (S^{*,1}, \dots, S^{*,d})$. Then we have the following.

Lemma 5.2.2. *Suppose that $\hat{\phi} = \{\hat{\phi}_t = (\hat{\phi}_t^1, \dots, \hat{\phi}_t^d), t \in [0, T]\}$ is a d -dimensional process such that*

- (i) $\hat{\phi} : [0, T] \times \Omega \rightarrow \mathbb{R}^d$ is $(\mathcal{B}_T \times \mathcal{F}_T)$ -measurable where $\hat{\phi}(t, \omega) = \hat{\phi}_t(\omega)$ for $t \in [0, T]$ and $\omega \in \Omega$;
- (ii) $\hat{\phi}$ is adapted, i.e., $\hat{\phi}_t \in \mathcal{F}_t$ for each $t \in [0, T]$;
- (iii) (a) $\int_0^T \left| \sum_{i=1}^d \hat{\phi}_t^i S_t^{*,i} (\mu_t^i - r_t) \right| dt < \infty$ P -a.s.,
 (b) $\int_0^T \left| \sum_{i=1}^d \hat{\phi}_t^i S_t^{*,i} \sigma_t^i \right|^2 dt < \infty$ P -a.s.

Let $G^*(\hat{\phi}) = \{G_t^*(\hat{\phi}), t \in [0, T]\}$ be a continuous, adapted process such that P -a.s., for all $t \in [0, T]$,

$$\begin{aligned} G_t^*(\hat{\phi}) &= \int_0^t \left(\sum_{i=1}^d \hat{\phi}_s^i S_s^{*,i} (\mu_s^i - r_s) \right) ds \\ &\quad + \int_0^t \left(\sum_{i=1}^d \hat{\phi}_s^i S_s^{*,i} \sigma_s^i \right) \cdot dW_s. \end{aligned}$$

Given a constant V_0 , let

$$\phi_t^0 = V_0 + G_t^*(\hat{\phi}) - \hat{\phi}_t \cdot \hat{S}_t^*, \quad t \in [0, T].$$

Then,

$$\phi = (\phi^0, \hat{\phi}^1, \dots, \hat{\phi}^d)$$

is a self-financing trading strategy with discounted value process given by

$$V_t^*(\phi) = V_0 + G_t^*(\hat{\phi}), \quad t \in [0, T].$$

Proof. It follows from the measurability properties of $\hat{\phi}$ and $G^*(\hat{\phi})$ that ϕ satisfies the first two measurability properties (i) and (ii) required of a trading strategy. Furthermore, the integrability conditions (iii)(b) and (iii)(c) required of a trading strategy are implied by the conditions imposed on $\hat{\phi}$. Note for this that since B is a continuous strictly positive process, each path is bounded away from zero and infinity and so the conditions in (iii) above, imposed on $\hat{\phi}$, hold if and only if they hold with S in place of S^* .

For the integrability condition (iii)(a) required of a trading strategy ϕ , we note that by the definition of ϕ^0 , for each $t \in [0, T]$,

$$\begin{aligned} \phi_t \cdot S_t &= B_t(\phi_t \cdot S_t^*) = B_t(\phi_t^0 + \hat{\phi}_t \cdot \hat{S}_t^*) \\ &= B_t(V_0 + G_t^*(\hat{\phi})), \end{aligned}$$

and so P -a.s., for all $t \in [0, T]$,

$$\int_0^t |\phi_s \cdot S_s| r_s ds \leq \sup_{s \in [0, T]} |B_s(V_0 + G_s^*(\hat{\phi}))| \left(\int_0^T r_s ds \right) < \infty,$$

by the integrability assumptions on r and since $\{B_s(V_0 + G_s^*(\hat{\phi})), s \in [0, T]\}$ is continuous and hence bounded pathwise. Thus, ϕ is a trading strategy. The value process for ϕ is given by

$$V_t(\phi) = \phi_t \cdot S_t = B_t(V_0 + G_t^*(\hat{\phi})), \quad t \in [0, T],$$

and the discounted value process is given by

$$V_t^*(\phi) = V_0 + G_t^*(\hat{\phi}), \quad t \in [0, T],$$

which is continuous and adapted. From the definition of $G^*(\hat{\phi})$ and the fact that $\hat{\phi}^i = \phi^i$ for $i = 1, \dots, d$, it follows that $V^*(\phi)$ satisfies (5.16). Hence, by Lemma 5.2.1, ϕ is self-financing. $\square$

An *arbitrage opportunity* is a self-financing trading strategy ϕ such that

$$V_0(\phi) = 0, \quad V_T(\phi) \geq 0, \quad \text{and} \quad P(V_T(\phi) > 0) > 0. \quad (5.18)$$

Note that the same definition results if the value process $V(\phi)$ is replaced by the discounted value process $V^*(\phi)$ in (5.18) above.

5.3. First Fundamental Theorem of Asset Pricing

Based on the first fundamental theorem of asset pricing for the finite market model described in Chapter 3, one might be tempted to conjecture the following (false) “folk” theorem: “In the multi-dimensional Black-Scholes model, there exists a risk neutral probability if and only if there are no arbitrage opportunities.” To obtain a rigorous version of this statement requires refinement of the notion of no arbitrage and careful specification of what one means by a risk neutral probability.

In the generality of the multi-dimensional Black-Scholes model considered here, the appropriate notion of risk neutral probability involves a change of probability such that, under a probability measure equivalent to P , the discounted asset price process is a *local martingale*. Recall from Appendix C that given a probability measure Q defined on $(\Omega, \mathcal{F})$, an m -dimensional process (where m is a positive integer), $M = \{M_t, t \in [0, T]\}$ defined on the filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$, is a *local martingale* under Q if M is an adapted process and there is a sequence of stopping times $\{\tau_k\}_{k=1}^\infty$ such that $\tau_k \leq \tau_{k+1}$ for each k , $\lim_{k \rightarrow \infty} \tau_k = \infty$ Q -a.s., and for each k ,

$$M^k \equiv \{M_{t \wedge \tau_k}, \mathcal{F}_t, t \in [0, T]\}$$

is an m -dimensional martingale under Q (i.e., each component of M^k is a one-dimensional martingale under Q). Further recall from Chapter 4 that two probability measures Q and $\tilde{Q}$ on $(\Omega, \mathcal{F})$ are equivalent (or mutually absolutely continuous) provided for each $A \in \mathcal{F}$,

$$Q(A) = 0 \quad \text{if and only if} \quad \tilde{Q}(A) = 0; \quad (5.19)$$

i.e., Q and $\tilde{Q}$ have the same sets of probability zero.

Definition 5.3.1. An *equivalent local martingale measure* (abbreviated as *ELMM*) is a probability measure P^* on $(\Omega, \mathcal{F})$ such that P^* is equivalent to P and $\{S_t^*, \mathcal{F}_t, t \in [0, T]\}$ is a local martingale under P^* .

Remark. In the context of this chapter, we shall also refer to an equivalent local martingale measure as a *risk neutral probability*.

The following terminology and notation is needed to define an appropriate notion of no arbitrage for the multi-dimensional Black-Scholes model where the asset price processes are given by an Itô process. For this, recall that we say a random variable is bounded if it is bounded P -a.s. and (in)equalities between random variables hold P -a.s. In particular, two random variables are considered to be equal if they are equal P -a.s. (Sometimes we shall include the P -a.s. for emphasis, but not always.)

Definition 5.3.2. *A tame strategy is a self-financing trading strategy ϕ for which there is a constant $c > 0$ such that P -a.s.,*

$$V_t^*(\phi) \geq -c \quad \text{for all } t \in [0, T].$$

Let

$$L = \{V_T^*(\phi) : \phi \text{ is a tame strategy and } V_0(\phi) = 0 \text{ } P\text{-a.s.}\}.$$

Thus, L is the set of discounted terminal values for portfolios that are achievable using tame strategies with zero initial value. We let D denote the set of bounded random variables that can be dominated by an element of L . That is, D is the set of bounded random variables such that there exists a tame strategy with an initial value of zero whose terminal discounted value is at least as large as that of the given random variable. If there is a non-negative random variable X in D satisfying $P(X > 0) > 0$, then there is an arbitrage opportunity using a tame strategy. A slightly more general notion of arbitrage opportunity is needed to yield a satisfactory first fundamental theorem. This is provided by considering the closure of D . Let $\bar{D}$ denote the closure of D in the space $L^\infty(\Omega, \mathcal{F}, P)$ of bounded, real-valued random variables defined on $(\Omega, \mathcal{F}, P)$ endowed with its usual essential supremum norm $\|\cdot\|_\infty$.

Definition 5.3.3. *The multi-dimensional Black Scholes model satisfies the “No Free Lunch with Vanishing Risk” (NFLVR) condition if and only if the only non-negative random variable in $\bar{D}$ is the zero random variable.*

The condition NFLVR implies that there is no sequence of tame strategies $\{\phi^n\}$ with zero initial values such that the discounted final payoffs satisfy $V_T^*(\phi^n) \geq -\frac{1}{n}$ P -a.s. for each n , and $Y \equiv \liminf_{n \rightarrow \infty} V_T^*(\phi^n)$ satisfies $P(Y > 0) > 0$.

The following is a statement of the *first fundamental theorem of asset pricing* in the context of our multi-dimensional Black-Scholes model. A proof of this result in a somewhat more general context (for locally bounded semimartingales) can be found in Delbaen and Schachermayer [14] (cf. Corollary 1.2 of Section 4). Their proof uses heavy machinery from stochastic processes and functional analysis. We refer the interested reader to that paper for the proof and also for a description of related forms of the result and for some

of the history associated with this topic. (In fact, the “only if” part of the theorem can be proved using the results on equivalent local martingale measures derived in Section 5.4 below. See the Exercises at the end of this chapter.)

Theorem 5.3.4. (*First Fundamental Theorem of Asset Pricing*) *An equivalent local martingale measure exists for the multi-dimensional Black-Scholes model if and only if there is “No Free Lunch with Vanishing Risk” (NFLVR).*

Our main interest in the preceding result is that it provides motivation for restricting to models for which there is at least one equivalent local martingale measure. In the next section we investigate the form of such measures. For later use, we say that the multi-dimensional Black-Scholes model is *viable* if and only if it satisfies the condition of “No Free Lunch with Vanishing Risk”. By the theorem above, this may be expressed by requiring that there is at least one equivalent local martingale measure.

5.4. Form of Equivalent Local Martingale Measures

In the remaining sections of this chapter, since we shall be dealing with more than one probability measure on $(\Omega, \mathcal{F})$, when taking expectations we shall explicitly indicate the relevant probability measure. Recall that we are assuming that $\mathcal{F} = \mathcal{F}_T$.

In the first theorem below, we characterize the probability measures on $(\Omega, \mathcal{F})$ that are equivalent to P .

Theorem 5.4.1. (*Equivalent Probability Measures*) *Q is a probability measure equivalent to P on $(\Omega, \mathcal{F})$ if and only if there is an n -dimensional, adapted process $\theta = \{\theta_t, t \in [0, T]\}$ satisfying*

- (i) $\theta : [0, T] \times \Omega \rightarrow \mathbb{R}^n$ is $(\mathcal{B}_T \times \mathcal{F}_T)$ -measurable, where $\theta(t, \omega) = \theta_t(\omega)$ for $t \in [0, T]$ and $\omega \in \Omega$;
- (ii) $\int_0^T |\theta_t|^2 dt < \infty$ P -a.s.,

such that

$$\Lambda_t = \exp \left(- \int_0^t \theta_s \cdot dW_s - \frac{1}{2} \int_0^t |\theta_s|^2 ds \right), \quad t \in [0, T], \quad (5.20)$$

defines a continuous martingale under P , and

$$Q(A) = E^P[1_A \Lambda_T] \quad \text{for all } A \in \mathcal{F}. \quad (5.21)$$

Remark. As usual, to ensure that Λ is well defined and has continuous paths everywhere, on an exceptional P -null set where $\int_0^T |\theta_t|^2 dt$ may not be finite, we define the integrals in the exponent of Λ_t to be zero for all t , and then Λ_t is one for all $t \in [0, T]$. A well-known sufficient condition for

the process Λ to be a martingale under P given that θ is an n -dimensional process satisfying conditions (i)–(ii) of Theorem 5.4.1 is called *Novikov's criterion*. This criterion requires that

$$E^P \left[\exp \left(\frac{1}{2} \int_0^T |\theta_t|^2 dt \right) \right] < \infty. \quad (5.22)$$

Proof. If θ is an n -dimensional adapted process satisfying conditions (i)–(ii) of the theorem, then the stochastic integral process $\{\int_0^t \theta_s \cdot dW_s, t \in [0, T]\}$ is well defined as a continuous local martingale under P . As described in the remark above, up to indistinguishability, $\{\int_0^t |\theta_s|^2 ds, t \in [0, T]\}$ defines a continuous, adapted process. Hence, Λ can be defined as a continuous, adapted process that satisfies $\Lambda_t > 0$ for all $t \in [0, T]$. In fact, under P , by applying Itô's formula to the Itô process

$$X_t = -\frac{1}{2} \int_0^t |\theta_s|^2 ds - \int_0^t \theta_s \cdot dW_s, \quad t \in [0, T]$$

and the function $f : \mathbb{R} \rightarrow (0, \infty)$ given by $f(x) = \exp(x)$, we see that P -a.s., for all $t \in [0, T]$,

$$\begin{aligned} \Lambda_t &= \exp(X_t) \\ &= 1 + \int_0^t \Lambda_s dX_s + \frac{1}{2} \int_0^t \Lambda_s d[X]_s \\ &= 1 - \int_0^t \Lambda_s \theta_s \cdot dW_s - \frac{1}{2} \int_0^t \Lambda_s |\theta_s|^2 ds \\ &\quad + \frac{1}{2} \int_0^t \Lambda_s |\theta_s|^2 ds \\ &= 1 - \int_0^t \Lambda_s \theta_s \cdot dW_s. \end{aligned} \quad (5.23)$$

Since Λ is a continuous, adapted process, $\Lambda\theta = \{\Lambda_t\theta_t, t \in [0, T]\}$ is adapted and satisfies the same conditions (i)–(ii) as θ ; hence the stochastic integral in the last line above is well defined as a continuous local martingale under P , and so Λ is a continuous local martingale under P . If in fact Λ is a martingale under P , then

$$E^P[\Lambda_T] = E^P[\Lambda_0] = 1$$

and (5.21) defines a probability measure Q on $(\Omega, \mathcal{F})$ and, since $\Lambda_T > 0$, this probability measure is equivalent to P . This completes the proof of the “if” part of the theorem.

To prove the “only if” part of the theorem, suppose that Q is a probability measure on $(\Omega, \mathcal{F})$ that is equivalent to P . Let

$$\zeta = \frac{dQ}{dP} \quad \text{on } \mathcal{F}; \quad (5.24)$$

i.e., ζ is the Radon-Nikodym derivative of Q with respect to P . (As with conditional expectations, such a random variable is uniquely defined only up to P -null sets.) Thus, ζ is a non-negative $\mathcal{F}_T$ -measurable (recall that $\mathcal{F}_T = \mathcal{F}$) random variable such that

$$Q(A) = E^P[\zeta 1_A] \quad \text{for each } A \in \mathcal{F}. \quad (5.25)$$

In particular, $E^P[\zeta] = Q(\Omega) = 1$ and so ζ is integrable with respect to P . Furthermore, since Q is *equivalent* to P , we have $\zeta > 0$ P -a.s. We define a martingale $\Lambda = \{\Lambda_t, t \in [0, T]\}$ under P by setting

$$\Lambda_t = E^P[\zeta \mid \mathcal{F}_t] \quad \text{for } t \in [0, T].$$

Since, for each $t \in [0, T]$, the conditional expectation above is defined only up to a P -null set and every martingale with respect to the Brownian filtration $\{\mathcal{F}_t\}$ under P has a continuous modification, without loss of generality, we may and do assume that all of the paths of the martingale Λ are continuous. Note that since ζ is measurable with respect to $\mathcal{F} = \mathcal{F}_T$, we have $\Lambda_T = \zeta$ P -a.s. Also, P -a.s.,

$$\Lambda_0 = E^P[\zeta \mid \mathcal{F}_0] = E^P[\zeta] = 1,$$

and since $\zeta \geq 0$ P -a.s., we have for each fixed $t \in [0, T]$,

$$\Lambda_t = E^P[\zeta \mid \mathcal{F}_t] \geq 0 \quad P\text{-a.s.}$$

In fact we can show that P -a.s., $\Lambda_t > 0$ for all $t \in [0, T]$, as follows.

Consider the stopping time

$$\tau = \inf\{t \in [0, T] : \Lambda_t = 0\},$$

where the infimum of the empty set is defined to be ∞ . Then by Doob's stopping theorem for martingales (cf. Appendix C),

$$E^P[\Lambda_{T \wedge \tau}] = E^P[\Lambda_0] = 1.$$

Splitting up the expectation on the left according to whether $\tau = \infty$ or $\tau \leq T$, we have

$$E^P[\Lambda_T 1_{\{\tau = \infty\}}] + E^P[\Lambda_\tau 1_{\{\tau \leq T\}}] = 1.$$

The second expectation above is zero since $\Lambda_\tau = 0$ on $\{\tau \leq T\}$. Hence,

$$E^P[\Lambda_T 1_{\{\tau = \infty\}}] = 1 = E^P[\Lambda_0] = E^P[\Lambda_T],$$

where we have used the martingale property of Λ to obtain the last equality. By subtraction we find that

$$E^P[\Lambda_T 1_{\{\tau \leq T\}}] = 0,$$

and hence since $\Lambda_T = \zeta$ P -a.s., we have

$$Q(\tau \leq T) = E^P[\zeta 1_{\{\tau \leq T\}}] = 0;$$

and since P is equivalent to Q , it follows that

$$P(\tau \leq T) = 0.$$

Hence, since $\Lambda_0 = 1 > 0$ P -a.s. and Λ has continuous paths, it follows that P -a.s.,

$$\Lambda_t > 0 \quad \text{for all } t \in [0, T], \quad (5.26)$$

as desired.

We now apply the martingale representation theorem to Λ (cf. Theorem D.6.1 of Appendix D). Since, under P , Λ is a continuous martingale with respect to the standard filtration $\{\mathcal{F}_t\}$ associated with the n -dimensional Brownian motion W , by the martingale representation theorem, there is an adapted n -dimensional process $\eta = \{\eta_t, t \in [0, T]\}$ such that $\eta : [0, T] \times \Omega \rightarrow \mathbb{R}^n$ is $(\mathcal{B}_T \times \mathcal{F}_T)$ -measurable, $\int_0^T |\eta_s|^2 ds < \infty$ P -a.s., and

$$\Lambda_t = \Lambda_0 + \int_0^t \eta_s \cdot dW_s \quad \text{for all } t \in [0, T], \quad P\text{-a.s.}$$

Let N be a P -null set such that (5.26) holds everywhere on $\Omega \setminus N$. Define

$$\theta_t(\omega) = \begin{cases} -\frac{\eta_t(\omega)}{\Lambda_t(\omega)} & \text{for all } t \in [0, T], \omega \in \Omega \setminus N, \\ 0 & \text{for all } t \in [0, T], \omega \in N. \end{cases} \quad (5.27)$$

Then, P -a.s.,

$$-\Lambda_t \theta_t = \eta_t \quad \text{for all } t \in [0, T].$$

By the continuity properties of Λ and the fact that (5.26) holds on $\Omega \setminus N$, the process θ inherits the properties of η ; i.e., $\theta = \{\theta_t, t \in [0, T]\}$ is an adapted process where $\theta : [0, T] \times \Omega \rightarrow \mathbb{R}^n$ is $(\mathcal{B}_T \times \mathcal{F}_T)$ -measurable and $\int_0^T |\theta_t|^2 dt < \infty$ P -a.s. Consequently, the stochastic integral process

$$\int_0^t \theta_s \cdot dW_s = \sum_{i=1}^n \int_0^t \theta_s^i dW_s^i, \quad t \in [0, T],$$

is well defined as a continuous local martingale under P (cf. Appendix D). Let N' denote a P -null set such that $\int_0^T |\theta_s|^2 ds < \infty$ on $\Omega \setminus N'$. Define a continuous adapted process $\rho = \{\rho_t, t \in [0, T]\}$ by

$$\rho_t = \begin{cases} \exp\left(-\int_0^t \theta_s \cdot dW_s - \frac{1}{2} \int_0^t |\theta_s|^2 ds\right), & \text{for } t \in [0, T], \text{ on } \Omega \setminus N', \\ 1, & \text{for } t \in [0, T], \text{ on } N'. \end{cases}$$

Then, $\rho_0 = 1$, and by Itô's formula, ρ satisfies the following stochastic equation P -a.s.:

$$\rho_t = 1 - \int_0^t \rho_s \theta_s \cdot dW_s, \quad t \in [0, T]. \quad (5.28)$$

On the other hand, $\Lambda_0 = 1$ P -a.s., and Λ satisfies the following stochastic equation P -a.s.:

$$\Lambda_t = 1 + \int_0^t \eta_s \cdot dW_s = 1 - \int_0^t \Lambda_s \theta_s \cdot dW_s, \quad t \in [0, T]. \quad (5.29)$$

Thus, Λ and ρ satisfy the same stochastic equation with the same initial condition. We will show that in fact P -a.s.,

$$\rho_t = \Lambda_t \quad \text{for all } t \in [0, T]. \quad (5.30)$$

Indeed, since P -a.s., $\Lambda_t > 0$ and $\rho_t > 0$ for all $t \in [0, T]$, one can apply Itô's formula (cf. Section D.4 in Appendix D) to the two-dimensional Itô process (Λ, ρ) and the function $f : (0, \infty)^2 \rightarrow \mathbb{R}$ given by $f(x, y) = x/y$ to yield P -a.s. for all $t \in [0, T]$,

$$\begin{aligned} \frac{\Lambda_t}{\rho_t} &= \frac{\Lambda_0}{\rho_0} + \int_0^t \rho_s^{-1} d\Lambda_s - \int_0^t \rho_s^{-2} \Lambda_s d\rho_s \\ &\quad + \frac{1}{2} \left\{ \int_0^t \frac{2}{\rho_s^3} \Lambda_s d[\rho]_s - 2 \int_0^t \rho_s^{-2} d\langle \Lambda, \rho \rangle_s \right\} \\ &= \frac{\Lambda_0}{\rho_0} - \int_0^t \rho_s^{-1} \Lambda_s \theta_s \cdot dW_s + \int_0^t \rho_s^{-2} \Lambda_s \rho_s \theta_s \cdot dW_s \\ &\quad + \int_0^t \rho_s^{-3} \Lambda_s \rho_s^2 |\theta_s|^2 ds - \int_0^t \rho_s^{-2} \rho_s \Lambda_s |\theta_s|^2 ds \\ &= 1. \end{aligned}$$

Here, $[\rho]$ denotes the quadratic variation process for ρ which, using the stochastic differential equation satisfied by ρ , can be seen to be given P -a.s. by

$$[\rho]_t = \int_0^t \rho_s^2 |\theta_s|^2 ds, \quad t \in [0, T],$$

and $\langle \Lambda, \rho \rangle$ is the mutual variation (or covariation) process for Λ and ρ which, using the stochastic differential equations satisfied by Λ and ρ , can be seen to be given P -a.s. by

$$\langle \Lambda, \rho \rangle_t = \int_0^t \rho_s \Lambda_s |\theta_s|^2 ds, \quad t \in [0, T].$$

To summarize, we have shown that P -a.s.,

$$\Lambda_t = \rho_t = \exp \left(- \int_0^t \theta_s \cdot dW_s - \frac{1}{2} \int_0^t |\theta_s|^2 ds \right) \quad \text{for all } t \in [0, T], \quad (5.31)$$

where Λ is a continuous martingale with respect to P . Thus, P -a.s.,

$$\frac{dQ}{dP} = \zeta = \Lambda_T = \exp \left(- \int_0^T \theta_t \cdot dW_t - \frac{1}{2} \int_0^T |\theta_t|^2 dt \right). \quad (5.32)$$

□

The following lemma provides a useful characterization of local martingales under a probability measure equivalent to P . A proof of this can be found for example in Chung and Williams [11], Theorem 9.7.

Lemma 5.4.2. *Suppose that Q is a probability measure on $(\Omega, \mathcal{F})$ that is equivalent to P . Let $\theta = \{\theta_t, t \in [0, T]\}$ and $\Lambda = \{\Lambda_t, t \in [0, T]\}$ be as specified in Theorem 5.4.1. Then, a real-valued, right continuous adapted process $M = \{M_t, t \in [0, T]\}$ is a local martingale under Q if and only if $\{\Lambda_t M_t, t \in [0, T]\}$ is a local martingale under P .*

We now characterize the equivalent local martingale measures.

Theorem 5.4.3. *(Equivalent Local Martingale Measures) Q is an equivalent local martingale measure if and only if there is an n -dimensional, adapted process $\theta = \{\theta_t, t \in [0, T]\}$ such that*

- (i) θ satisfies conditions (i)–(ii) of Theorem 5.4.1,
- (ii) (5.20) defines a continuous martingale under P , and Q is given by (5.21),
- (iii) P -a.s.,

$$\mu_t^i - r_t - \sigma_t^i \cdot \theta_t = 0 \quad \text{for a.e. } t \in [0, T], \quad i = 1, \dots, d. \quad (5.33)$$

Remark. Here a.e. denotes “almost everywhere” with respect to Lebesgue measure on $[0, T]$. A process $\theta = \{\theta_t, t \in [0, T]\}$ satisfying all of the conditions in Theorem 5.4.3 is called a *market price of risk*. In general it need not be unique.

Proof. For a proof of the “only if” part of the theorem, suppose that Q is an equivalent local martingale measure. Since Q is equivalent to P , there is a process θ and an associated martingale $\Lambda = \{\Lambda_t, t \in [0, T]\}$ as described in Theorem 5.4.1, so that Λ_T yields the Radon-Nikodym derivative of Q with respect to P . By the local martingale property, S^* is a local martingale under Q . In fact, S^* has continuous paths, and so it is a continuous local martingale under Q . It follows from Lemma 5.4.2 that for each $i = 1, \dots, d$, $\Lambda S^{*,i} = \{\Lambda_t S_t^{*,i}, t \in [0, T]\}$ is a local martingale under P , and hence $\Lambda S^* = (\Lambda S^{*,1}, \dots, \Lambda S^{*,d})$ is a continuous d -dimensional local martingale under P .

For each $i = 1, \dots, d$, we apply Itô’s formula to the two-dimensional Itô process $(S^{*,i}, \Lambda)$ under P , and the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $f(x, y) = xy$, $(x, y) \in \mathbb{R}^2$. Using the stochastic differential equations (5.11) and (5.23)

satisfied by $S^{*,i}$ and Λ , we have P -a.s. for all $t \in [0, T]$:

$$\begin{aligned}
\Lambda_t S_t^{*,i} &= \Lambda_0 S_0^{*,i} + \int_0^t \Lambda_s dS_s^{*,i} + \int_0^t S_s^{*,i} d\Lambda_s + \langle \Lambda, S^{*,i} \rangle_t \\
&= S_0^{*,i} + \int_0^t \Lambda_s S_s^{*,i} (\mu_s^i - r_s) ds + \int_0^t \Lambda_s S_s^{*,i} \sigma_s^i \cdot dW_s \\
&\quad - \int_0^t S_s^{*,i} \Lambda_s \theta_s \cdot dW_s - \int_0^t \Lambda_s S_s^{*,i} \sigma_s^i \cdot \theta_s ds \\
&= 1 + \int_0^t \Lambda_s S_s^{*,i} (\mu_s^i - r_s - \sigma_s^i \cdot \theta_s) ds \\
&\quad + \int_0^t \Lambda_s S_s^{*,i} (\sigma_s^i - \theta_s) \cdot dW_s.
\end{aligned} \tag{5.34}$$

The equation above has a continuous local martingale (under P) on the left side. On the right side there are a sum of a continuous, adapted process whose paths are P -a.s. of bounded variation on $[0, T]$ and a continuous local martingale (under P) defined by the stochastic integral with respect to W . Rearranging to isolate the bounded variation process, we obtain a continuous local martingale (under P) equal to a continuous, adapted process whose paths are P -a.s. of bounded variation. This can happen only if both sides are P -a.s. identically constant (cf. [11], Corollary 4.5). Hence the continuous, adapted process whose paths are P -a.s. of bounded variation is P -a.s. identically constant. Thus, P -a.s.,

$$\int_0^t \Lambda_s S_s^{*,i} (\mu_s^i - r_s - \sigma_s^i \cdot \theta_s) ds = 0 \quad \text{for all } t \in [0, T].$$

This implies that P -a.s.,

$$\Lambda_t S_t^{*,i} (\mu_t^i - r_t - \sigma_t^i \cdot \theta_t) = 0 \quad \text{for a.e. } t \in [0, T]. \tag{5.35}$$

Now since P -a.s., $\Lambda_t > 0$ and $S_t^{*,i} > 0$ for all $t \in [0, T]$, it follows that P -a.s., θ satisfies (5.33). This completes the proof of the “only if” part of the theorem.

For a proof of the “if” part of the theorem, let θ be an n -dimensional, adapted process satisfying conditions (i)–(iii) of the theorem. Let $\Lambda = \{\Lambda_t, t \in [0, T]\}$ and Q be defined as in (5.20) and (5.21), respectively. Then, by Theorem 5.4.1, Q is equivalent to P , and it suffices to show that S^* is a local martingale under Q . By Lemma 5.4.2, for the latter, it suffices to show that $\{\Lambda_t S_t^{*,i}, t \in [0, T]\}$ is a local martingale under P for each $i = 1, \dots, d$. In the same manner as for (5.34), since $(S^{*,i}, \Lambda)$ is a two-dimensional Itô process under P , by applying Itô’s formula to this process and the function $f(x, y) = xy$, $(x, y) \in \mathbb{R}^2$, we have that for $i = 1, \dots, d$, (5.34) holds P -a.s.

for all $t \in [0, T]$. Then using (5.33), this simplifies to

$$\Lambda_t S_t^{*,i} = 1 + \int_0^t \Lambda_s S_s^{*,i} (\sigma_s^i - \theta_s^i) \cdot dW_s.$$

Thus, we have that $\Lambda S^{*,i}$ is a constant plus a stochastic integral with respect to the Brownian motion W under P , and hence it is a local martingale under P . It then follows from Lemma 5.4.2 that $S^{*,i}$ is a local martingale under Q , as desired. $\square$

Corollary 5.4.4. *Suppose that Q is an equivalent local martingale measure and that θ is an n -dimensional process as described in Theorem 5.4.3. Define the n -dimensional, continuous adapted process $\tilde{W} = \{\tilde{W}_t, t \in [0, T]\}$ such that*

$$\tilde{W}_t = W_t + \int_0^t \theta_s ds, \quad t \in [0, T]. \quad (5.36)$$

Then, on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q)$, $\tilde{W}$ is a standard n -dimensional Brownian motion martingale and Q -a.s., for $i = 1, \dots, d$,

$$S_t^{*,i} = S_0^{*,i} + \int_0^t S_s^{*,i} \sigma_s^i \cdot d\tilde{W}_s, \quad t \in [0, T]. \quad (5.37)$$

Furthermore, for any self-financing trading strategy ϕ , the discounted value process $V^(\phi)$ is a continuous local martingale under Q that Q -a.s. satisfies*

$$V_t^*(\phi) = V_0(\phi) + \int_0^t \left(\sum_{i=1}^d \phi_s^i S_s^{*,i} \sigma_s^i \right) \cdot d\tilde{W}_s, \quad t \in [0, T]. \quad (5.38)$$

Remark. As usual, in the definition of $\tilde{W}$, we use a continuous, adapted representative of $\{\int_0^t \theta_s ds, t \in [0, T]\}$. The integrals in (5.37) and (5.38) are all well defined as stochastic integrals (yielding continuous local martingales) under Q .

Proof. By Girsanov's theorem (cf. Appendix D, Theorem D.5.1), $\tilde{W}$ is an n -dimensional Brownian motion martingale on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q)$. Now, under P , $\tilde{W}$ is an n -dimensional Itô process; and on substituting for W in terms of $\tilde{W}$ and θ in the stochastic differential equation (5.11) satisfied by S^* and substituting the relation (5.33), we obtain P -a.s., for all $t \in [0, T]$ and $i = 1, \dots, d$,

$$\begin{aligned} S_t^{*,i} &= S_0^{*,i} + \int_0^t S_s^{*,i} (\mu_s^i - r_s - \sigma_s^i \cdot \theta_s) ds + \int_0^t S_s^{*,i} \sigma_s^i \cdot d\tilde{W}_s \\ &= S_0^{*,i} + \int_0^t S_s^{*,i} \sigma_s^i \cdot d\tilde{W}_s. \end{aligned}$$

Note that, P -a.s., the integral $\int_0^t \sigma_s^i \cdot \theta_s ds$ is finite for all $t \in [0, T]$ by the Cauchy-Schwarz inequality and the integral assumptions on σ and θ . Also,

the integral with respect to $\tilde{W}$ can be defined as a P -almost sure limit of integrals with respect to $\tilde{W}$ against simple predictable integrands. Since Q is equivalent to P , the above relation holds Q -a.s., and under Q , the integral with respect to $\tilde{W}$ defines a continuous local martingale stochastic integral process with respect to the n -dimensional Brownian motion martingale $\tilde{W}$. This establishes (5.37).

By Lemma 5.2.1, for a self-financing trading strategy ϕ , (5.16) holds P -a.s., where on substituting the relation (5.33) and the definition of $\tilde{W}$, we obtain P -a.s., for all $t \in [0, T]$,

$$\begin{aligned} V_t^*(\phi) &= V_0(\phi) + \int_0^t \left(\sum_{i=1}^d \phi_s^i S_s^{*,i} \sigma_s^i \cdot \theta_s \right) ds \\ &\quad + \int_0^t \left(\sum_{i=1}^d \phi_s^i S_s^{*,i} \sigma_s^i \right) \cdot dW_s, \\ &= \int_0^t \left(\sum_{i=1}^d \phi_s^i S_s^{*,i} \sigma_s^i \right) \cdot d\tilde{W}_s. \end{aligned}$$

By similar reasoning to that in the paragraph above, since P is equivalent to Q , the above relation also holds Q -a.s., where the last expression is well defined as a continuous local martingale stochastic integral process under Q . $\square$

Theorems 5.4.1 and 5.4.3 can be combined with the remark following Theorem 5.4.1 to yield the following result. For this, let $\mathbf{1}$ denote the d -dimensional vector of all ones.

Corollary 5.4.5. *Assume that $d = n$ and P -a.s., σ_t is invertible for a.e. $t \in [0, T]$. Suppose that $\theta = \{\theta_t, t \in [0, T]\}$ is an n -dimensional, adapted process such that P -a.s.,*

$$\sigma_t \theta_t = \mu_t - r_t \mathbf{1}, \quad \text{for a.e. } t \in [0, T]. \quad (5.39)$$

Further suppose that θ satisfies conditions (i)–(ii) of Theorem 5.4.1, as well as Novikov's criterion (5.22). Then, $\Lambda = \{\Lambda_t, t \in [0, T]\}$ defined by (5.20) in Theorem 5.4.1 is a martingale and Q defined by (5.21) is the unique equivalent local martingale measure.

Proof. The only detail requiring verification is the uniqueness of the ELMM. This follows from the fact that given an equivalent local martingale measure Q , there is always a θ as described in Theorem 5.4.3; and when $d = n$ and P -a.s., σ_t is invertible for a.e. $t \in [0, T]$, this relation determines $\theta : [0, T] \times \Omega \rightarrow \mathbb{R}^n$ up to sets of $(\lambda \times P)$ -measure zero, where λ denotes Lebesgue measure on

$[0, T]$. This suffices for uniqueness up to indistinguishability of the associated process Λ given by (5.20). Hence Q (cf. (5.21)) is uniquely determined by σ, μ and r . $\square$

5.5. Second Fundamental Theorem of Asset Pricing

Recall from the end of Section 5.3 that the multi-dimensional Black-Scholes model is *viable* if there is at least one equivalent local martingale measure (ELMM). We henceforth assume that the model is viable and let P^* be an equivalent local martingale measure.

Paralleling the definition of admissible strategies for the simple Black-Scholes model given in Chapter 4, we make the following definition for the multi-dimensional model considered here. In contrast to the situation for the simple model, here there may be more than one equivalent local martingale measure and so the set of admissible strategies can depend on the choice of that measure. Of course, if there is a unique ELMM, then the definition of admissible strategy is unequivocal.

An *admissible strategy* (relative to P^*) is a self-financing trading strategy ϕ such that the discounted value process $\{V_t^*(\phi), t \in [0, T]\}$ is a martingale under the equivalent local martingale measure P^* .

Given the form (5.38) of $V^*(\phi)$ under the ELMM P^* , a sufficient condition for a self-financing trading strategy ϕ to be admissible is that

$$E^{P^*} \left[\int_0^T \left| \sum_{i=1}^d \phi_s^i S_s^{*,i} \sigma_s^i \right|^2 ds \right] < \infty. \quad (5.40)$$

In this case, the discounted value process $V^*(\phi)$ will be an L^2 -martingale under P^* (cf. Appendix D).

A *European contingent claim* is represented by an $\mathcal{F}_T$ -measurable random variable X . We denote the discounted value of a European contingent claim X by

$$X^* = X/B_T = X/S_T^0.$$

A *replicating (or hedging) strategy* (relative to P^*) for a European contingent claim X is an admissible strategy (relative to P^*) such that $V_T(\phi) = X$, P^* -a.s.

Theorem 5.5.1. (*Second Fundamental Theorem of Asset Pricing*) Suppose there exists an equivalent local martingale measure P^* . Then the following two conditions are equivalent.

- (i) P^* is the unique equivalent local martingale measure.

- (ii) Every $\mathcal{F}_T$ -measurable random variable X satisfying $E^{P^*}[|X^*|] < \infty$ is replicable (relative to P^*).

Proof. (i) implies (ii): This part of the proof uses a general result taken from Jacod and Yor [25]. We state the consequence of that result here without proof and refer the reader to that paper for details.

Suppose that P^* is the unique equivalent local martingale measure. Let θ and $\tilde{W}$ be as described in Corollary 5.4.4 with P^* in place of Q there. Let X be an $\mathcal{F}_T$ -measurable random variable satisfying $E^{P^*}[|X^*|] < \infty$. Then, the conditional expectations

$$E^{P^*}[X^*|\mathcal{F}_t], \quad t \in [0, T],$$

define a martingale under P^* , and we may choose a modification that has right continuous paths with finite left limits. We denote such a representative by $M = \{M_t, t \in [0, T]\}$. It then follows from a result in the general theory of martingale representations (cf. Jacod and Yor [25], Proposition 2.4) that there is an adapted d -dimensional process $\eta = \{\eta_t, t \in [0, T]\}$ satisfying

- (a) $\eta : [0, T] \times \Omega \rightarrow \mathbb{R}^d$ is $(\mathcal{B}_T \times \mathcal{F}_T)$ -measurable where $\eta(t, \omega) = \eta_t(\omega)$ for $t \in [0, T]$ and $\omega \in \Omega$, and
(b) P^* -a.s.,

$$\sum_{i=1}^d \int_0^T |\eta_t^i S_t^{*,i} \sigma_t^i|^2 dt < \infty, \quad (5.41)$$

and such that P^* -a.s., for all $t \in [0, T]$,

$$\begin{aligned} M_t &= M_0 + \int_0^t \eta_s \cdot dS_s^* \\ &= M_0 + \sum_{i=1}^d \int_0^t \eta_s^i dS_s^{*,i} \\ &= M_0 + \sum_{i=1}^d \int_0^t \eta_s^i S_s^{*,i} \sigma_s^i \cdot d\tilde{W}_s \\ &= M_0 + \int_0^t \sum_{i=1}^d (\eta_s^i S_s^{*,i} \sigma_s^i) \cdot d\tilde{W}_s. \end{aligned} \quad (5.42)$$

In other words, we can represent M as a stochastic integral with respect to the local martingale process S^* . It follows from this representation that M has continuous paths P^* -a.s. Note that by the Cauchy-Schwarz inequality, for each $t \in [0, T]$,

$$\left| \sum_{i=1}^d \eta_t^i S_t^{*,i} \sigma_t^i \right|^2 \leq d \sum_{i=1}^d |\eta_t^i S_t^{*,i} \sigma_t^i|^2. \quad (5.43)$$

It follows from this and the conditions satisfied by η , and since P^* is equivalent to P , that $\hat{\phi} = \eta$ satisfies conditions (i)–(ii) and (iii)(b) of Lemma 5.2.2. In fact, condition (iii)(a) there also holds because θ satisfies the conditions in Theorem 5.4.3. In particular, P -a.s.,

$$\begin{aligned} \int_0^T \left| \sum_{i=1}^d \eta_t^i S_t^{*,i} (\mu_t^i - r_t) \right| dt &= \int_0^T \left| \sum_{i=1}^d \eta_t^i S_t^{*,i} \sigma_t^i \cdot \theta_t \right| dt \\ &\leq \left(\int_0^T \left| \sum_{i=1}^d \eta_t^i S_t^{*,i} \sigma_t^i \right|^2 dt \right)^{\frac{1}{2}} \left(\int_0^T |\theta_t|^2 dt \right)^{\frac{1}{2}} \\ &< \infty, \end{aligned}$$

by (5.43), (5.41), and condition (ii) of Theorem 5.4.1 satisfied by θ . On substituting the expression (5.36) for $\tilde{W}$ in (5.42) and using (5.33), it follows, since the integrals are all well defined, that the last line in (5.42) is equal P^* -a.s., and hence P -a.s., for all $t \in [0, T]$ to

$$M_0 + \int_0^t \left(\sum_{i=1}^d \eta_s^i S_s^{*,i} (\mu_s^i - r_s) \right) ds + \int_0^t \left(\sum_{i=1}^d \eta_s^i S_s^{*,i} \sigma_s^i \right) \cdot dW_s. \quad (5.44)$$

Here the last integral is a stochastic integral with respect to W that defines a continuous local martingale under P . It then follows from Lemma 5.2.2 that $\hat{\phi} = \eta$ can be extended, by the addition of another component, to a self-financing trading strategy ϕ with discounted value process $V^*(\phi)$ satisfying P -a.s.,

$$V_t^*(\phi) = M_t \quad \text{for all } t \in [0, T]. \quad (5.45)$$

In particular, since P is equivalent to P^* , we have P^* -a.s.,

$$V_T^*(\phi) = M_T = X^*, \quad (5.46)$$

and hence $V_T(\phi) = X$ P^* -a.s. Since (5.45) holds and M is a martingale under P^* , it follows that ϕ is a replicating strategy for X .

(ii) *implies (i)*: Assume that condition (ii) in the theorem holds and that Q is some equivalent local martingale measure for the model. We need to show that $Q = P^*$. Fix an event $A \in \mathcal{F}_T$. Let $X = 1_A S_T^0$, which is an $\mathcal{F}_T$ -measurable random variable and satisfies $E^{P^*}[|X^*|] = E^{P^*}[1_A] \leq 1 < \infty$. By our hypothesis (ii), X is replicable (relative to P^*). So there exists an admissible strategy ϕ (relative to P^*) such that $V_T(\phi) = X$, P^* -a.s. By the admissibility of ϕ , the process $\{V_t^*(\phi), t \in [0, T]\}$ is a martingale under P^* .

By the replicating property of ϕ and the martingale property of $V^*(\phi)$ under P^* , we have for each $t \in [0, T]$, P^* -a.s.,

$$V_t^*(\phi) = E^{P^*}[V_T^*(\phi) \mid \mathcal{F}_t] = E^{P^*}[1_A \mid \mathcal{F}_t] \geq 0.$$

However, $V^*(\phi)$ has continuous paths, and so we can interchange the order of the qualifiers $t \in [0, T]$ and P^* -a.s. to conclude that P^* -a.s.,

$$V_t^*(\phi) \geq 0 \quad \text{for all } t \in [0, T]. \quad (5.47)$$

Since P^* is equivalent to Q (as they are both equivalent to P), it follows that (5.47) also holds Q -a.s.

Using the replicating property of ϕ and the martingale property of $V^*(\phi)$ under P^* again, we have

$$\begin{aligned} P^*(A) &= E^{P^*}[1_A] = E^{P^*}[V_T^*(\phi)] \\ &= E^{P^*}[V_0^*(\phi)] = V_0(\phi). \end{aligned} \quad (5.48)$$

Now, by Corollary 5.4.4, since Q is an ELMM, $V^*(\phi)$ is a continuous *local* martingale under Q , and so there is a localizing sequence of stopping times $\{\tau_k\}$ such that $\tau_k \leq \tau_{k+1}$ for all k , $\tau_k \rightarrow \infty$ Q -a.s. as $k \rightarrow \infty$, and $\{V_{t \wedge \tau_k}^*(\phi), t \in [0, T]\}$ is a *martingale* under Q for each k . In particular, $V_T^*(\phi) = \lim_{k \rightarrow \infty} V_{T \wedge \tau_k}^*(\phi)$ Q -a.s. Since (5.47) holds Q -a.s., we have for each k , $V_{T \wedge \tau_k}^*(\phi) \geq 0$ Q -a.s. Thus, we may apply Fatou's lemma to conclude that

$$\begin{aligned} Q(A) &= E^Q[V_T^*(\phi)] \\ &= E^Q \left[\lim_{k \rightarrow \infty} V_{T \wedge \tau_k}^*(\phi) \right] \\ &\leq \liminf_{k \rightarrow \infty} E^Q[V_{T \wedge \tau_k}^*(\phi)] \\ &= \liminf_{k \rightarrow \infty} E^Q[V_0^*(\phi)] \\ &= V_0(\phi) \\ &= P^*(A). \end{aligned}$$

Here we have used Fatou's lemma for the inequality, the martingale property of the stopped process for the fourth line, and (5.48) for the last equality.

Since $A \in \mathcal{F}_T$ was arbitrary, it follows that $P^*(A) \geq Q(A)$ for *every* $A \in \mathcal{F}_T$. To complete the proof that (ii) implies (i) by contradiction, suppose we have the *strict* inequality $P^*(\tilde{A}) > Q(\tilde{A})$ for some $\tilde{A} \in \mathcal{F}_T$. Then, applying this inequality and the previous one (with $A = \tilde{A}^c = \Omega \setminus \tilde{A}$), we obtain

$$1 = P^*(\Omega) = P^*(\tilde{A}) + P^*(\tilde{A}^c) > Q(\tilde{A}) + Q(\tilde{A}^c) = 1.$$

This contradiction shows that $P^*(A) = Q(A)$ for all $A \in \mathcal{F}_T$. Hence $Q = P^*$, and we have established (i). $\square$

If condition (ii) of Theorem 5.5.1 is satisfied, then we say that the multi-dimensional Black-Scholes model is *complete*. By that same theorem, under our assumption that the model is viable, completeness is equivalent to uniqueness of the equivalent local martingale measure P^* .

To further characterize complete markets in the context of the multi-dimensional Black-Scholes model, we shall assume that the number of stocks d is not greater than the dimension n of the underlying Brownian motion W . One may argue that if this assumption does not hold, then the market model has a certain redundancy that can be eliminated by replacing the more than n stocks by n mutual funds that are time and realization dependent linear combinations of the stocks. We refer the reader to Karatzas and Shreve [28], Remark 1.4.10, for an explanation of this rationale.

Theorem 5.5.2. *Suppose that there exists an equivalent local martingale measure P^* and that $d \leq n$. Then the following two conditions are equivalent.*

- (i) *The multi-dimensional Black-Scholes model is complete.*
- (ii) *We have $d = n$, and P -a.s., the $d \times d$ matrix σ_t is invertible for a.e. $t \in [0, T]$.*

Remark. Note that since P is equivalent to P^* , in condition (ii) one could just as well use P^* -a.s. in place of P -a.s.

Proof. We give a nearly complete proof of this result. The main omission is in the proof that (i) implies (ii), where a construction of a suitable Borel measurable function ψ is assumed (we provide a reference for this result).

(ii) *implies (i):* Assume that $d = n$ and that P -a.s., σ_t is invertible for a.e. $t \in [0, T]$. Let Q be any equivalent local martingale measure for the model. Then by Theorem 5.4.3, there is a market price of risk $\theta = \{\theta_t, t \in [0, T]\}$ such that $\frac{dQ}{dP} = \Lambda_T$ where Λ_T is given by (5.20), and θ satisfies (5.33). Using the invertibility assumption on σ , it follows that P -a.s.,

$$\theta_t = \sigma_t^{-1}(\mu_t - r_t \mathbf{1}) \quad \text{for a.e. } t \in [0, T], \quad (5.49)$$

where σ_t^{-1} denotes the inverse of σ_t when it exists. This relation uniquely determines (up to almost sure equivalence) the integrals used in defining Λ_T , and hence Λ_T is uniquely determined by the model parameters σ , μ , and r . Since Λ_T determines Q from P , it follows that the measure Q is uniquely determined by the model parameters and P . Consequently, there is only one equivalent local martingale measure Q , and so we must have $Q = P^*$. Thus, there is a unique equivalent local martingale measure, and by the second fundamental theorem of asset pricing (Theorem 5.5.1), the model is complete.

(i) implies (ii): Suppose that (i) holds. In particular, suppose that (ii) of Theorem 5.5.1 holds. Consider the space $\mathcal{M}$ of $d \times n$ matrices endowed with the norm $|A| = (\sum_{i=1}^d \sum_{j=1}^n |A_{ij}|^2)^{\frac{1}{2}}$ for $A \in \mathcal{M}$. Viewing an element, A , of this space as a linear function $A : \mathbb{R}^n \rightarrow \mathbb{R}^d$, let $\mathcal{N}(A)$ denote the null space of A ; i.e., $\mathcal{N}(A) = \{x \in \mathbb{R}^n : Ax = 0\}$. It is well known from linear algebra that $\mathcal{N}(A) = \mathcal{R}(A')^\perp$, where the latter denotes the orthogonal complement in $\mathbb{R}^n$ of the range $\mathcal{R}(A')$ of the (linear) transpose $A' : \mathbb{R}^d \rightarrow \mathbb{R}^n$ of A . One can show (see [28], Lemma 1.6.9 for a proof) that there is a bounded, Borel measurable function $\psi : \mathcal{M} \rightarrow \mathbb{R}^n$ such that for each $A \in \mathcal{M}$, $\psi(A) \in \mathcal{N}(A)$ and $\psi(A) \neq 0$ if $\mathcal{N}(A) \neq \{0\}$. Then the n -dimensional process κ , defined by the composition $\kappa_t = \psi(\sigma_t)$, $t \in [0, T]$, is bounded and adapted, $\kappa : [0, T] \times \Omega \rightarrow \mathbb{R}^n$ is $(\mathcal{B}_T \times \mathcal{F}_T)$ -measurable where $\kappa(t, \omega) = \kappa_t(\omega)$, and for each $t \in [0, T]$, $\omega \in \Omega$, $\kappa_t(\omega) \in \mathcal{N}(\sigma_t(\omega))$ and $\kappa_t(\omega) \neq \{0\}$ if $\mathcal{N}(\sigma_t(\omega)) \neq \{0\}$. Let a market price of risk θ and process $\tilde{W}$ associated with P^* be as described in Corollary 5.4.4 with P^* in place of Q there. Since κ is bounded, the stochastic integral process

$$\int_0^t \kappa_s \cdot d\tilde{W}_s, \quad t \in [0, T],$$

defines a continuous L^2 -martingale under P^* . Let

$$X = S_T^0 \left(1 + \int_0^T \kappa_t \cdot d\tilde{W}_t \right).$$

Then, X is $\mathcal{F}_T$ -measurable and

$$E^{P^*}[|X^*|] \leq E^{P^*} \left[1 + \left| \int_0^T \kappa_t \cdot d\tilde{W}_t \right| \right] < \infty.$$

We also have by the martingale property of the stochastic integral that

$$E^{P^*}[X^*] = 1.$$

Since we are assuming that the model is complete, there is an admissible strategy (under P^*) that replicates X . We denote this strategy by ϕ . Then, $\{V_t^*(\phi), t \in [0, T]\}$ is a martingale under P^* and $V_T^*(\phi) = X^*$ P^* -a.s. Consequently,

$$V_0(\phi) = V_0^*(\phi) = E^{P^*}[V_T^*(\phi)] = E^{P^*}[X^*] = 1.$$

Since ϕ is self-financing, we also have by Corollary 5.4.4 that P^* -a.s. for all $t \in [0, T]$,

$$V_t^*(\phi) = V_0(\phi) + \int_0^t \left(\sum_{i=1}^d \phi_s^i S_s^{*,i} \sigma_s^i \right) \cdot d\tilde{W}_s.$$

Equating the values of $V_T^*(\phi)$ we have P^* -a.s.,

$$\begin{aligned} X^* &= 1 + \int_0^T \kappa_t \cdot d\tilde{W}_t \\ &= V_T^*(\phi) = 1 + \int_0^T \left(\sum_{i=1}^d \phi_t^i S_t^{*,i} \sigma_t^i \right) \cdot d\tilde{W}_t. \end{aligned}$$

Hence, upon subtracting we obtain, P^* -a.s.,

$$\int_0^T \left(\kappa_t - \sum_{i=1}^d \phi_t^i S_t^{*,i} \sigma_t^i \right) \cdot d\tilde{W}_t = 0.$$

Taking the conditional expectation $E^{P^*}[\cdot | \mathcal{F}_t]$ of both sides and using the martingale property of the stochastic integral under P^* , we obtain P^* -a.s., for all $t \in [0, T]$,

$$\int_0^t \left(\kappa_s - \sum_{i=1}^d \phi_s^i S_s^{*,i} \sigma_s^i \right) \cdot d\tilde{W}_s = 0.$$

The left member above defines a continuous martingale under P^* . Since the right member is zero, the quadratic variation process of this martingale must equal the zero process and so, P^* -a.s., for all $t \in [0, T]$,

$$\int_0^t \left| \kappa_s - \sum_{i=1}^d \phi_s^i S_s^{*,i} \sigma_s^i \right|^2 ds = 0.$$

It follows from this that P^* -a.s.,

$$\kappa_t = \sum_{i=1}^d \phi_t^i S_t^{*,i} \sigma_t^i = \sigma_t' \eta_t, \quad \text{for a.e. } t \in [0, T],$$

where $\eta_t^i = \phi_t^i S_t^{*,i}$ for $i = 1, \dots, d$, $t \in [0, T]$. Thus, P^* -a.s., $\kappa_t \in \mathcal{R}(\sigma_t')$ for a.e. $t \in [0, T]$. But by construction, $\kappa_t \in \mathcal{N}(\sigma_t) = \mathcal{R}(\sigma_t')^\perp$, and so it follows that P^* -a.s., $\kappa_t = 0$ for a.e. $t \in [0, T]$. Since $\kappa_t \neq 0$ whenever $\mathcal{N}(\sigma_t) \neq \{0\}$, it follows that P^* -a.s., $\mathcal{N}(\sigma_t) = \{0\}$ for a.e. $t \in [0, T]$. Hence, since we assumed that $d \leq n$, we must have that $d = n$ and that P^* -a.s., σ_t is invertible for a.e. $t \in [0, T]$. As P^* is equivalent to P , the conclusion that (ii) holds follows. $\square$

5.6. Pricing European Contingent Claims

In this section, we assume that the multi-dimensional Black-Scholes model, which we view as the primary market model, is viable and complete. In this case there is a unique equivalent local martingale measure which we denote by P^* . Let θ and $\tilde{W}$ be as defined in Corollary 5.4.4 with P^* in place of Q there.

Consider a European contingent claim represented by $X \in \mathcal{F}_T$ and satisfying $E^{P^*}[|X^*|] < \infty$. We shall determine the unique arbitrage free price process for this claim. Our treatment is similar to that in Section 4.5. As usual, we need a notion of admissible strategy for trading in the stocks, money market and contingent claim.

We assume that any given price process $C = \{C_t, t \in [0, T]\}$ for the European contingent claim is a right continuous, adapted process, where C_t represents the price of the European contingent claim at time t . We suppose that for a given price process C , there is a set of admissible strategies Ψ_C for trading in the stocks, money market and contingent claim. At a minimum, an element ψ of Ψ_C should be a $(d+2)$ -dimensional adapted process $\psi = \{\psi_t = (\phi_t, \gamma_t), t \in [0, T]\}$ where $\phi = \{\phi_t, t \in [0, T]\}$ is a $(d+1)$ -dimensional adapted process representing the holdings over the time interval $[0, T]$ in the primary market assets (the money market instrument and the stocks), and $\gamma = \{\gamma_t, t \in [0, T]\}$ is a one-dimensional adapted process representing the holdings over $[0, T]$ in the contingent claim. We shall also want elements of Ψ_C to be self-financing in an appropriate way.

To show uniqueness of the arbitrage free price process, we do not want to place unnecessary restrictions on C just so that it can be used as an integrator (in defining self-financing trading strategies). In fact, to show uniqueness, we shall need only the reasonable assumption that rather simple strategies of the following type are admissible: those where one buys or sells one contingent claim, or does nothing, at some deterministic time t^* , and then holds that position in the claim until time T , but where one may trade as usual in the stocks and money market over $[t^*, T]$. More precisely, we assume that strategies of the following form are in Ψ_C :

$$\psi_t(\omega) = 1_A(\omega)(\tilde{\phi}_t^0(\omega) + \tilde{\delta}_t(\omega), \tilde{\phi}_t^1(\omega), \dots, \tilde{\phi}_t^d(\omega), \tilde{\gamma}_t(\omega)), \quad t \in [t^*, T], \quad (5.50)$$

and $\psi_t(\omega) = 0$ for $t \in [0, t^*)$, for all $\omega \in \Omega$, where t^* is a fixed time in $[0, T]$, $A \in \mathcal{F}_{t^*}$, $\tilde{\phi} = \{(\tilde{\phi}_t^0, \tilde{\phi}_t^1, \dots, \tilde{\phi}_t^d), t \in [0, T]\}$ is an admissible self-financing trading strategy in the primary (money market and stocks) market, $\tilde{\gamma}_t = +1$ for all $t \in [t^*, T]$ or $\tilde{\gamma}_t = -1$ for all $t \in [t^*, T]$, and $\tilde{\delta}_t = \tilde{\delta}_{t^*}$ for all $t \in [t^*, T]$ is chosen such that $\tilde{\phi}_{t^*}^0 \cdot S_{t^*} + \tilde{\delta}_{t^*} S_{t^*}^0 + \tilde{\gamma}_{t^*} C_{t^*} = 0$. Under such a strategy, the investor does nothing prior to time t^* . On A^c , the investor continues to do nothing for the remaining interval $[t^*, T]$. On A , if $\tilde{\gamma}_{t^*} \equiv +1$, at t^* the investor buys one contingent claim for C_{t^*} and invests $-C_{t^*}$ in the money market instrument and stocks according to $(\tilde{\phi}_t^0 + \tilde{\delta}_t, \tilde{\phi}_t^1, \dots, \tilde{\phi}_t^d)$ for $t \in [t^*, T]$. That is, on A , for $t \in [t^*, T]$, the investor follows the strategy $\tilde{\phi}$ in the primary market, and, in addition, purchases $\tilde{\delta}_{t^*}$ shares of the money market instrument at time t^* and holds those until time T . On A , if $\tilde{\gamma}_{t^*} \equiv -1$, instead of buying one contingent claim at time t^* , the investor sells one contingent claim at t^* and invests the proceeds C_{t^*} according to $(\tilde{\phi}_t^0 +$

$\tilde{\delta}_t, \tilde{\phi}_t^1, \dots, \tilde{\phi}_t^d$ for $t \in [t^*, T]$. Since the value of the strategy ψ is zero at time t^* and $\tilde{\delta}_t$ and $\tilde{\gamma}_t$ do not vary with $t \in [t^*, T]$, the strategy ψ will be naturally self-financing, since $\tilde{\phi}$ is assumed to have this property.

To show existence of the arbitrage free price process, we will prove below that when the price process C is a continuous modification of

$$S_t^0 E^{P^*}[X^* | \mathcal{F}_t], \quad t \in [0, T], \quad (5.51)$$

it is an arbitrage free price process. (A priori, one might expect that C should only be right continuous. However, it turns out that any right continuous modification of the above process is P^* -a.s. continuous as it is indistinguishable from the value of a replicating strategy for the European contingent claim.) For the existence proof, we need constraints on the admissible strategies similar to what we had in the case of the primary market. Note that if ϕ^* is a replicating strategy for X , then by the martingale property of $V^*(\phi^*)$ under P^* , we have $V_t^*(\phi^*) = E^{P^*}[X^* | \mathcal{F}_t]$ P^* -a.s. for each $t \in [0, T]$. So $C_t \equiv S_t^0 V_t^*(\phi^*)$, $t \in [0, T]$, is a continuous modification of (5.51). By Corollary 5.4.4 we have P^* -a.s.,

$$V_t^*(\phi^*) = V_0(\phi) + \int_0^t \left(\sum_{i=1}^d \phi_s^{*,i} S_s^{*,i} \sigma_s^i \right) \cdot d\tilde{W}_s, \quad t \in [0, T]. \quad (5.52)$$

Thus, given

$$C_t^* = C_t / S_t^0 = V_t^*(\phi^*), \quad t \in [0, T], \quad (5.53)$$

it is natural to let the class of admissible strategies Ψ_C be the set of $(d+2)$ -dimensional processes $\psi = (\phi, \gamma)$ such that

- (i) $\psi : [0, T] \times \Omega \rightarrow \mathbb{R}^{d+2}$ is $(\mathcal{B}_T \times \mathcal{F}_T)$ -measurable, where $\psi(t, \omega) = \psi_t(\omega)$ for all $t \in [0, T]$ and $\omega \in \Omega$;
- (ii) ψ is adapted, i.e., $\psi_t \in \mathcal{F}_t$ for each $t \in [0, T]$;
- (iii) ϕ is a $(d+1)$ -dimensional process that is a trading strategy in the primary market consisting of the money market instrument and stocks;
- (iv) γ is a one-dimensional process satisfying P^* -a.s.,

$$\int_0^T \gamma_s^2 \left| \sum_{i=1}^d \phi_s^{*,i} S_s^{*,i} \sigma_s^i \right|^2 ds < \infty; \quad (5.54)$$

- (v) the value process $\{V_t(\psi), t \in [0, T]\}$ defined by

$$V_t(\psi) \equiv \phi_t \cdot S_t + \gamma_t C_t, \quad t \in [0, T], \quad (5.55)$$

is a continuous, adapted process such that the discounted value process $\{V_t^*(\psi), t \in [0, T]\}$ defined by

$$V_t^*(\psi) = V_t(\psi) / S_t^0, \quad t \in [0, T], \quad (5.56)$$

is a continuous martingale under P^* that satisfies P^* -a.s.,

$$\begin{aligned} V_t^*(\psi) &= V_0(\psi) + \int_0^t \left(\sum_{i=1}^d \phi_s^i S_s^{*,i} \sigma_s^i \right) \cdot d\tilde{W}_s \\ &\quad + \int_0^t \gamma_s \left(\sum_{i=1}^d \phi_s^{*,i} S_s^{*,i} \sigma_s^i \right) \cdot d\tilde{W}_s. \end{aligned} \quad (5.57)$$

Properties (i)–(iv) above ensure that the stochastic integrals appearing in (5.57), which is a form of self-financing condition on ψ , are well defined as continuous local martingales under P^* . Note that since S^* and C^* (given by (5.53)) are already martingales under P^* , the martingale condition on $V^*(\psi)$ amounts to some control on the size of ψ to rule out doubling strategies. One can verify that strategies of the form (5.50) satisfy the above properties (i)–(v) when C^* is given by (5.53).

Given a price process C for the contingent claim, an *arbitrage opportunity* in the stocks-money market-contingent claim market is an admissible strategy $\psi \in \Psi_C$ such that $V_0(\psi) = 0$, $V_T(\psi) \geq 0$ and $P^*(V_T(\psi) > 0) > 0$. (Since P and P^* are equivalent, we could just as easily have used P in place of P^* in the last inequality.)

In the following theorem, uniqueness means uniqueness up to indistinguishability of stochastic processes.

Theorem 5.6.1. *The unique arbitrage free price process for the European contingent claim X is a continuous modification of*

$$\{S_t^0 E^{P^*}[X^* | \mathcal{F}_t], \quad t \in [0, T]\}. \quad (5.58)$$

Proof. Consider a price process $\{C_t, t \in [0, T]\}$ for the European contingent claim. Since the primary market is assumed to be viable and complete, there is a replicating strategy ϕ^* for X , and its discounted value process $\{V_t^*(\phi^*), t \in [0, T]\}$ being a martingale under P^* with terminal value X^* must satisfy for each $t \in [0, T]$, P^* -a.s.,

$$V_t^*(\phi^*) = E^{P^*}[V_T^*(\phi^*) | \mathcal{F}_t] = E^{P^*}[X^* | \mathcal{F}_t].$$

Thus, the undiscounted value process $V(\phi^*)$, which is assumed to be continuous and adapted, is a continuous modification of $\{S_t^0 E^{P^*}[X^* | \mathcal{F}_t], t \in [0, T]\}$.

Suppose, for a proof of uniqueness by contradiction, that

$$P^*(C_t \neq V_t(\phi^*) \text{ for some } t \in [0, T]) > 0.$$

Since the process C is assumed to be right continuous and $V(\phi^*)$ is continuous, there exists a $t^* \in [0, T]$ such that $P^*(C_{t^*} \neq V_{t^*}(\phi^*)) > 0$. Let $A = \{\omega \in \Omega : C_{t^*}(\omega) > V_{t^*}(\phi^*)(\omega)\}$ and $\bar{A} = \{\omega \in \Omega : C_{t^*}(\omega) < V_{t^*}(\phi^*)(\omega)\}$.

Then either $P^*(A) > 0$ or $P^*(\tilde{A}) > 0$. First suppose that $P^*(A) > 0$. Define $\psi = \{(\phi_t, \gamma_t), t \in [0, T]\}$ by

$$\psi_t(\omega) = \begin{cases} 0, & t < t^*, \\ 0, & t \geq t^*, \omega \in A^c, \\ \left(\phi_t^{*,0} + \frac{C_{t^*} - V_{t^*}(\phi^*)}{S_{t^*}^0}, \phi_t^{*,1}, \dots, \phi_t^{*,d}, -1 \right) (\omega), & t \geq t^*, \omega \in A. \end{cases}$$

The value of the portfolio represented by ψ_t is zero for $t \leq t^*$, and so the value at time zero is zero. Moreover, the value of ψ at time T is zero on A^c . Also, since ϕ^* is a replicating strategy for X , on A , the value of ψ at time T is $\left(\frac{C_{t^*} - V_{t^*}(\phi^*)}{S_{t^*}^0} \right) S_T^0$, which is strictly positive. Since $P^*(A) > 0$, ψ is an arbitrage opportunity.

Similarly, if $P^*(\tilde{A}) > 0$, then ψ defined for $t \in [0, T]$, $\omega \in \Omega$ by

$$\psi_t(\omega) = 1_{\{t \geq t^*\}} 1_{\tilde{A}}(\omega) \left(-\phi_t^{*,0} + \frac{V_{t^*}(\phi^*) - C_{t^*}}{S_{t^*}^0}, -\phi_t^{*,1}, \dots, -\phi_t^{*,d}, 1 \right) (\omega), \quad (5.59)$$

is an arbitrage opportunity.

Thus, we have shown that the only possible arbitrage free price process (up to indistinguishability) is given by $C_t = V_t(\phi^*)$, $t \in [0, T]$. Next we show that this price process is arbitrage free. For a contradiction, suppose that with this price process there is $\psi \in \Psi_C$ such that $V_0(\psi) = 0$, $V_T(\psi) \geq 0$ and $P^*(V_T(\psi) > 0) > 0$. Now, by the admissibility assumptions on ψ , the discounted value process $V^*(\psi)$ is a martingale under P^* and so

$$0 = V_0^*(\psi) = E^{P^*}[V_T^*(\psi)] = E^{P^*} \left[\frac{V_T(\psi)}{S_T^0} \right] > 0, \quad (5.60)$$

a contradiction. Thus, there cannot be an arbitrage opportunity with this price process. $\square$

5.7. Incomplete Markets

Assuming the primary market model is viable, that is, there is at least one equivalent local martingale measure P^* , if the model is incomplete, then by the characterization afforded by the second fundamental theorem of asset pricing,

- (i) there is more than one equivalent local martingale measure (ELMM) and
- (ii) for any ELMM, P^* , there are European contingent claims whose discounted values are integrable under P^* but for which there is no replicating strategy.

Scrutiny of the proof of Theorem 5.6.1 reveals that if X is the value of a European contingent claim whose discounted value is integrable under some ELMM P^* and the European contingent claim is replicable relative to P^* , then a continuous modification of (5.58) is the unique arbitrage free price process for the European contingent claim. However, if X is not replicable relative to P^* , then, in a similar manner to that for finite market models, some additional criterion needs to be invoked to choose even an initial arbitrage free price for the European contingent claim. For further discussion of this, we refer the reader to Musiela and Rutkowski [32] and Karatzas and Shreve [28], for a start.

5.8. Exercises

In these exercises, we use the notation of the multi-dimensional Black-Scholes model.

1. The Black-Scholes model described in Chapter 4 can be thought of as a multi-dimensional Black-Scholes model with $d = n = 1$ and constant parameters μ, r, σ as described in Chapter 4. Show that in this case, the conditions placed on a trading strategy in Chapter 4 are equivalent to those described in this chapter.
2. Assume $d = n \geq 1$, μ, r, σ are constant processes and σ is invertible. Show that there is a unique equivalent local martingale measure and specify its form.
3. Using the results from Section 5.4, in particular, Corollary 5.4.4, prove the “only if” part of the first fundamental theorem of asset pricing. That is, prove that if there is an equivalent local martingale measure, then there is No Free Lunch with Vanishing Risk.
4. The following parameter choices for the multi-dimensional Black-Scholes model yield a simple stochastic volatility model. Assume $d = 1$, $n = 2$, β, r are constants, and the stock price process satisfies the equation

$$dS_t^1 = \beta S_t^1 dt + S_t^1 g(Z_t) dW_t^1, \quad (5.61)$$

where $g : (0, \infty) \rightarrow (0, \infty)$ is continuous and bounded below by a strictly positive constant, and Z is a continuous adapted process satisfying

$$dZ_t = \gamma Z_t dt + \delta Z_t dW_t^2, \quad (5.62)$$

γ, δ are constants, $\delta > 0$, and Z_0 is a strictly positive constant. In fact,

$$Z_t = Z_0 \exp \left(\gamma t + \delta W_t^2 - \frac{1}{2} \delta^2 t \right), \quad t \in [0, T],$$

is the unique continuous adapted process satisfying (5.62) given Z_0 . Show that this model is viable and it is not complete.