

Black-Scholes Model

In this chapter we consider a simple continuous (in both time and space) financial market model called the Black-Scholes model. This can be viewed as a continuous analogue of the binomial model. Indeed, it can be obtained as a limit of a sequence of binomial models (with appropriate parameters). We shall prove the famous Black-Scholes formula for pricing European call options. In 1973, Fischer Black and Myron Scholes [5] developed a formula for the valuation of European contingent claims based on a geometric Brownian motion model for the stock price process. Robert Merton [31] developed another method to derive the formula that turned out to have very wide applicability; he also generalized the formula in many directions. In 1997, this work was recognized with the award of the Nobel Prize in Economics to Myron Scholes and Robert Merton (Fischer Black had passed away in 1995).

We begin this chapter by defining the Black-Scholes model and the notion of admissible trading strategy. We derive the equivalent martingale measure for this model and show that every suitably integrable European contingent claim has a replicating strategy. We use this to find the arbitrage free price process for any suitably integrable European contingent claim. We apply this result to the pricing of a European call option, which results in the famous Black-Scholes option pricing formula. Using the solution of the Black-Scholes partial differential equation, we also exhibit a replicating strategy in this case. We then consider American contingent claims and derive the arbitrage free initial price for a suitably integrable American contingent claim. We illustrate this result for American call and put options.

In this and subsequent chapters, we shall use various notions and concepts from the theory of continuous time stochastic processes and stochastic integration. For the convenience of the reader, a summary of some relevant concepts and results is provided in Appendices C and D. The reader may find it helpful to briefly review the material in that appendix before proceeding further. For further details, the reader is encouraged to consult other texts such as Chung [9] and Chung and Williams [11], Karatzas and Shreve [27], Revuz and Yor [35], or the two-volume set by Rogers and D. Williams [37, 36].

Regarding notational conventions, in this and subsequent chapters, we continue to use those conventions mentioned at the beginning of Chapter 2. In particular, if X is a real-valued random variable defined on a probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{G}$ is a sub- σ -algebra of $\mathcal{F}$, we shall sometimes use the notation $X \in \mathcal{G}$ to indicate that X is $\mathcal{G}$ -measurable. In contrast to the situation for the discrete models considered in previous chapters where there were only trivial sets of probability zero, from here on, typically there can be many sets of probability zero. (In)equalities between two random variables will be considered to hold almost surely. This will sometimes be indicated explicitly for emphasis, but often we shall omit special mention of the almost sure nature of (in)equalities. In a similar vein, a random variable will be called bounded if it is bounded almost surely. Two stochastic processes will be considered equal if they are indistinguishable (cf. Appendices B and C). In the following, $\log$ denotes the natural logarithm and for a vector $x \in \mathbb{R}^d$, $|x|$ will denote the Euclidean norm of x given by $|x| = \left(\sum_{i=1}^d (x^i)^2\right)^{\frac{1}{2}}$.

4.1. Preliminaries

We consider a finite time interval $[0, T]$, for some $0 < T < \infty$, as the interval during which trading may take place. The Borel σ -algebra of subsets of $[0, T]$ will be denoted by $\mathcal{B}_T$.

We assume as given a complete probability space $(\Omega, \mathcal{F}, P)$ on which is defined a standard one-dimensional *Brownian motion* $W = \{W_t, t \in [0, T]\}$. Let $\{\mathcal{F}_t, t \in [0, T]\}$ denote the standard filtration generated by the Brownian motion W under P (cf. Appendix D, Section D.1). It is well known that this filtration is right continuous; i.e., for each $t \in [0, T)$, $\mathcal{F}_t = \mathcal{F}_{t+} \equiv \bigcap_{s \in (t, T]} \mathcal{F}_s$ (cf. Chung [9], Section 2.3, Theorem 4). All random variables considered in this chapter will be assumed to be defined on $(\Omega, \mathcal{F}_T)$, and so without loss of generality we assume that $\mathcal{F} = \mathcal{F}_T$. We shall frequently write $\{\mathcal{F}_t\}$ instead of the more cumbersome $\{\mathcal{F}_t, t \in [0, T]\}$. The filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ satisfies the usual conditions (cf. Appendix C). It is well known that under this assumption, every martingale has a modification

whose paths are all right continuous with finite left limits; cf. Chung [9], Theorem 3, page 29 and Corollary 1, page 26. In fact, since the filtration is generated by the Brownian motion W , every martingale has a continuous modification; cf. Theorems II.2.9, V.3.5 in Revuz and Yor [35]. Expectations with respect to P will be denoted by $E[\cdot]$ unless there is more than one probability measure under consideration, in which case we shall use $E^P[\cdot]$ in place of $E[\cdot]$.

4.2. Black-Scholes Model

Our Black-Scholes model has two assets, a (risky) stock with price process $S = \{S_t, t \in [0, T]\}$ and a (riskless) bond with price process $B = \{B_t, t \in [0, T]\}$. These processes are given by

$$S_t = S_0 \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right), \quad t \in [0, T], \quad (4.1)$$

$$B_t = e^{rt}, \quad t \in [0, T], \quad (4.2)$$

where $r \geq 0$, $\mu \in \mathbb{R}$, $\sigma > 0$, and $S_0 > 0$ is a positive constant. The parameter σ is called the *volatility*. The process S is frequently called a *geometric Brownian motion*. It is well known (cf. Chung and Williams [11], Theorem 6.2) that $\{e^{-\mu t} S_t, \mathcal{F}_t, t \in [0, T]\}$ is an L^2 -martingale under P . The processes S, B are continuous, adapted, and satisfy the following dynamic equations P -a.s.:

$$S_t = S_0 + \mu \int_0^t S_s ds + \sigma \int_0^t S_s dW_s, \quad t \in [0, T], \quad (4.3)$$

$$B_t = B_0 + r \int_0^t B_s ds, \quad t \in [0, T]. \quad (4.4)$$

We shall sometimes write these equations in the following differential form for convenience in performing calculations using Itô's formula (cf. Appendix D, Section D.4):

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad t \in [0, T], \quad (4.5)$$

$$dB_t = r B_t dt, \quad t \in [0, T]. \quad (4.6)$$

Remark. To see the connection with the binomial model, note that in the Black-Scholes model, $\log(S_t/S_0) = (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t$, a Brownian motion with drift, whereas in the binomial model, $\log(S_t/S_0) = \sum_{i=1}^t \xi_i$, which is a (possibly biased) random walk. It is well known that a sequence of random walks with appropriate rescaling and parameter values can be approximated by a Brownian motion with drift.

A *trading strategy* is a two-dimensional stochastic process $\phi = \{\phi_t = (\alpha_t, \beta_t), t \in [0, T]\}$ satisfying

- (i) $\phi : [0, T] \times \Omega \rightarrow \mathbb{R}^2$ is $(\mathcal{B}_T \times \mathcal{F}_T)$ -measurable where $\phi(t, \omega) = \phi_t(\omega)$ for each $t \in [0, T]$ and $\omega \in \Omega$;
- (ii) ϕ is adapted, i.e., $\phi_t \in \mathcal{F}_t$ for each $t \in [0, T]$;
- (iii) $\int_0^T \alpha_t^2 dt < \infty$ and $\int_0^T |\beta_t| dt < \infty$ P -a.s.

We sometimes abbreviate the description of conditions like (i) by saying that ϕ is $(\mathcal{B}_T \times \mathcal{F}_T)$ -measurable. The above conditions ensure that integrals of the form

$$\begin{aligned} \int_0^t \alpha_s dS_s &= \mu \int_0^t \alpha_s S_s ds + \sigma \int_0^t \alpha_s S_s dW_s, \\ \int_0^t \beta_s dB_s &= r \int_0^t \beta_s B_s ds \end{aligned}$$

are finite P -a.s. By defining the integrals to be identically zero on the exceptional null set, we may and do assume that they define continuous, adapted stochastic processes (see the definition of an Itô process in Appendix D). Here α_t is interpreted as the number of shares of stock held at time t , and β_t is interpreted as the number of units of the bond held at time t . The *value* at time t of the portfolio associated with ϕ is given by

$$V_t(\phi) = \alpha_t S_t + \beta_t B_t. \quad (4.7)$$

A trading strategy ϕ is said to be *self-financing* if $V(\phi) = \{V_t(\phi), t \in [0, T]\}$ is a continuous adapted process such that P -a.s.,

$$V_t(\phi) = V_0(\phi) + \int_0^t \alpha_s dS_s + \int_0^t \beta_s dB_s, \quad \text{for all } t \in [0, T], \quad (4.8)$$

or, in other words,

$$dV_t(\phi) = \alpha_t dS_t + \beta_t dB_t, \quad t \in [0, T]. \quad (4.9)$$

Thus, changes in the value of the portfolio result only from changes in the values of the assets so that there is no external infusion of capital and no spending of wealth.

Let ϕ be a trading strategy. The *discounted stock price process* and *discounted value process* associated with ϕ are given by

$$S_t^* = \frac{S_t}{B_t} = e^{-rt} S_t, \quad t \in [0, T], \quad (4.10)$$

$$V_t^*(\phi) = \frac{V_t(\phi)}{B_t} = e^{-rt} V_t(\phi), \quad t \in [0, T], \quad (4.11)$$

respectively. The pair (S, B^{-1}) is a two-dimensional Itô process where $B_t^{-1} = e^{-rt}$ is of bounded variation on $[0, T]$ and satisfies

$$dB_t^{-1} = -rB_t^{-1} dt, \quad t \in [0, T]. \quad (4.12)$$

By applying Itô's formula (cf. Appendix D, Section D.4) to this Itô process with the function $f(x, y) = xy$ for $x, y \in \mathbb{R}$, and using (4.5), (4.12), we have P -a.s. for all $t \in [0, T]$:

$$\begin{aligned} S_t^* &= S_0^* - r \int_0^t S_s^* ds + \int_0^t e^{-rs} dS_s \\ &= S_0^* + (\mu - r) \int_0^t S_s^* ds + \sigma \int_0^t S_s^* dW_s. \end{aligned} \quad (4.13)$$

Lemma 4.2.1. *A trading strategy ϕ is self-financing if and only if $V^*(\phi)$ is a continuous, adapted process such that P -a.s., for each $t \in [0, T]$,*

$$V_t^*(\phi) = V_0(\phi) + \int_0^t \alpha_s dS_s^*. \quad (4.14)$$

Here $\int_0^t \alpha_s dS_s^* = (\mu - r) \int_0^t \alpha_s S_s^* ds + \sigma \int_0^t \alpha_s S_s^* dW_s$.

Proof. In the following, for convenience, we use the differential forms of integral equations which hold P -a.s. However, each of the steps could be written in integral form to give a totally rigorous proof. The essential aspect is that we use the rules of stochastic calculus for manipulating differentials at each stage.

Note that $V(\phi)$ is continuous and adapted if and only if $V^*(\phi)$ is continuous and adapted. So it suffices to establish the equivalence of (4.14) with (4.8). Suppose that ϕ is a self-financing trading strategy. Then, by applying Itô's formula to the two-dimensional Itô process $(V(\phi), B^{-1})$ with the function $f(x, y) = xy$ for $x, y \in \mathbb{R}$, on using (4.11), (4.7), (4.9), and (4.12), we obtain P -a.s. for each $t \in [0, T]$,

$$\begin{aligned} dV_t^*(\phi) &= -re^{-rt}V_t(\phi)dt + e^{-rt}dV_t(\phi) \\ &= e^{-rt}(-r\alpha_t S_t dt - r\beta_t B_t dt + \alpha_t dS_t + \beta_t dB_t). \end{aligned}$$

Using (4.5), (4.6), and (4.13), the above yields

$$dV_t^*(\phi) = e^{-rt}\alpha_t((\mu - r)S_t dt + \sigma S_t dW_t) = \alpha_t dS_t^*, \quad (4.15)$$

for each $t \in [0, T]$, which proves that (4.14) holds.

Conversely, suppose that (4.14) holds. Then using (4.13), (4.6), and (4.7), we have P -a.s. for each $t \in [0, T]$,

$$\begin{aligned} dV_t^*(\phi) &= \alpha_t dS_t^* = e^{-rt}\alpha_t(-rS_t dt + dS_t) \\ &= e^{-rt}(-r\alpha_t S_t dt - r\beta_t B_t dt + \beta_t dB_t + \alpha_t dS_t) \\ &= e^{-rt}(-rV_t(\phi) dt + \alpha_t dS_t + \beta_t dB_t). \end{aligned}$$

Then, by applying Itô's formula to the Itô process $(V^*(\phi), B)$ and the function $f(x, y) = xy$ for $x, y \in \mathbb{R}$, we obtain

$$\begin{aligned} dV_t(\phi) &= d(e^{rt}V_t^*(\phi)) = re^{rt}V_t^*(\phi)dt + e^{rt}dV_t^*(\phi) \\ &= rV_t(\phi)dt + (-rV_t(\phi)dt + \alpha_t dS_t + \beta_t dB_t) \\ &= \alpha_t dS_t + \beta_t dB_t, \end{aligned}$$

and hence ϕ is self-financing. $\square$

A self-financing trading strategy ϕ is an *arbitrage opportunity* if

$$V_0(\phi) = 0, \quad V_T(\phi) \geq 0, \quad \text{and} \quad P(V_T(\phi) > 0) > 0. \quad (4.16)$$

An equivalent definition is obtained if the last term above is replaced by

$$E[V_T(\phi)] > 0,$$

since in the presence of the other conditions this is equivalent to $P(V_T(\phi) > 0) > 0$.

In contrast to the discrete time case, where we had only finitely many trading times and gains or losses are controlled by the initial investment, in continuous time, one can change one's strategy infinitely many times in a finite time interval and thereby obtain unbounded gains or losses by use of a doubling type of strategy. The following is a concrete example of such a strategy taken from Karatzas and Shreve [28].

Example. Consider the Black-Scholes model with $r = 0$, $\mu = 0$, and $\sigma = 1$. Then

$$B_t = 1 \quad \text{and} \quad dS_t = S_t dW_t, \quad t \in [0, T]. \quad (4.17)$$

Define

$$I(t) = \int_0^t \frac{1}{\sqrt{T-s}} dW_s, \quad \text{for all } 0 \leq t < T. \quad (4.18)$$

This defines a continuous, adapted stochastic integral process on $[0, T)$. The quadratic variation of I is given by

$$[I]_t = \int_0^t \frac{1}{T-s} ds = \log\left(\frac{T}{T-t}\right), \quad \text{for each } t < T. \quad (4.19)$$

Note that $[I]_0 = 0$, $[I]_t$ is increasing with t , and $[I]_t \rightarrow \infty$ as $t \rightarrow T$. It follows that I can be time changed to a Brownian motion (cf. Chung and Williams [11], Section 9.3) and in particular, for each $a > 0$ and

$$\tau_a \equiv \inf\{t \in [0, T] : I(t) = a\} \wedge T, \quad (4.20)$$

we have $0 < \tau_a < T$ P -a.s. Fix $a > 0$ and let N denote the P -null set on which $\tau_a \geq T$. Define $\phi = \{(\alpha_t, \beta_t), t \in [0, T]\}$ by setting for each $t \in [0, T]$:

$$\alpha_t = \begin{cases} (T-t)^{-\frac{1}{2}} S_t^{-1} 1_{\{t \leq \tau_a\}}, & \text{on } N, \\ 0, & \text{on } N^c, \end{cases} \quad (4.21)$$

$$\beta_t = \begin{cases} I(t \wedge \tau_a) - \alpha_t S_t, & \text{on } N, \\ 0, & \text{on } N^c. \end{cases} \quad (4.22)$$

As $\mathcal{F}_t$ contains the P -null sets for each $t \in [0, T]$, it is straightforward to verify that ϕ satisfies the properties required of a trading strategy. Since $B_t = 1$ for all t , the value process for ϕ satisfies the following P -a.s. for all $t \in [0, T]$:

$$V_t(\phi) = I(t \wedge \tau_a) = \int_0^{t \wedge \tau_a} \frac{1}{\sqrt{T-s}} dW_s \quad (4.23)$$

$$= \int_0^t \alpha_s S_s dW_s = \int_0^t \alpha_s dS_s, \quad (4.24)$$

where we have used the fact that S solves (4.5) with $\mu = 0$ and $\sigma = 1$. Since $r = 0$, it follows that (4.14) holds and hence ϕ is a self-financing trading strategy. Note that $V_0(\phi) = 0$ and P -a.s., $V_T(\phi) = I(\tau_a) = a > 0$ and so ϕ is an arbitrage opportunity.

There are several ways to add conditions to ϕ to rule out such arbitrage strategies. These usually amount to constraints on the size of integrals of ϕ with respect to (S, B) . Since pricing of contingent claims based on this model will use an equivalent martingale measure (or risk neutral probability), these conditions are often phrased in terms of such a probability measure. Some authors require that the associated value process is L^2 -bounded under an equivalent martingale measure, some require that the discounted value process is a martingale under such a measure, and some put integrability constraints directly on ϕ . Before discussing our conditions on ϕ , we introduce the equivalent martingale measure.

4.3. Equivalent Martingale Measure

Definition 4.3.1. Two probability measures, Q and $\tilde{Q}$, defined on $(\Omega, \mathcal{F})$ are equivalent (or mutually absolutely continuous) provided for each $A \in \mathcal{F}$,

$$Q(A) = 0 \quad \text{if and only if} \quad \tilde{Q}(A) = 0. \quad (4.25)$$

An equivalent martingale measure (abbreviated as EMM) is a probability measure P^* on $(\Omega, \mathcal{F})$ such that P^* is equivalent to P and $\{S_t^*, \mathcal{F}_t, t \in [0, T]\}$ is a martingale under P^* .

Remark. An equivalent martingale measure is sometimes also called a *risk neutral probability*. In this chapter, we shall use the terms interchangeably.

We wish to show that such an equivalent martingale measure P^* exists. For this note that

$$S_t^* = S_0 \exp \left(\left(\mu - r - \frac{1}{2}\sigma^2 \right) t + \sigma W_t \right) \quad (4.26)$$

and in particular, P -a.s.,

$$dS_t^* = (\mu - r)S_t^* dt + \sigma S_t^* dW_t, \quad t \in [0, T]. \quad (4.27)$$

The second term on the right in (4.27), when integrated, defines a martingale under P . We seek a probability measure P^* equivalent to P such that the discounted stock price process S^* satisfies a differential equation like (4.27) but without the drift term $(\mu - r)S_t^* dt$. For this, we use a simple form of the Girsanov transformation for changing the drift of a Brownian motion (cf. Appendix D, Theorem D.5.1). Let

$$\theta = \frac{\mu - r}{\sigma}, \quad (4.28)$$

$$\Lambda_t = \exp \left(-\theta W_t - \frac{1}{2}\theta^2 t \right), \quad t \in [0, T]. \quad (4.29)$$

Then it is well known (cf. [11], Theorem 6.2) that $\{\Lambda_t, \mathcal{F}_t, t \in [0, T]\}$ is a positive martingale under P . On $(\Omega, \mathcal{F})$, we define a new probability measure P^* so that

$$\frac{dP^*}{dP} = \Lambda_T \quad \text{on } \mathcal{F}; \quad (4.30)$$

i.e.,

$$P^*(A) = E^P [1_A \Lambda_T], \quad A \in \mathcal{F}, \quad (4.31)$$

where we have added the superscript P to the expectation E to emphasize that it is computed under the probability measure P . Similarly, we shall use E^{P^*} to denote the expectation operator under P^* . Note that since $\{\Lambda_t, \mathcal{F}_t, t \in [0, T]\}$ is a P -martingale, $P^*(\Omega) = E^P [\Lambda_T] = \Lambda_0 = 1$ and so P^* is indeed a probability measure on $(\Omega, \mathcal{F})$. Since $\Lambda_T > 0$, it follows that for each $A \in \mathcal{F}$, $P^*(A) = 0$ if and only if $P(A) = 0$. Thus, P^* is equivalent to P . All that remains is to show that the discounted stock price process S^* is a martingale under P^* . For this, let

$$\tilde{W}_t = W_t + \theta t, \quad t \in [0, T]. \quad (4.32)$$

By Girsanov's theorem (cf. Theorem D.5.1), $\{\tilde{W}_t, \mathcal{F}_t, t \in [0, T]\}$ is a standard Brownian motion and it is a martingale under P^* . Thus, for each $t \in [0, T]$, by (4.26) and (4.28), we have

$$\begin{aligned} S_t^* &= S_0 \exp \left(\left(\mu - r - \frac{1}{2}\sigma^2 \right) t + \sigma \left(\tilde{W}_t - \theta t \right) \right) \\ &= S_0 \exp \left(-\frac{1}{2}\sigma^2 t + \sigma \tilde{W}_t \right), \end{aligned} \quad (4.33)$$

and so P^* -a.s.,

$$dS_t^* = \sigma S_t^* d\tilde{W}_t, \quad t \in [0, T], \quad (4.34)$$

where $\tilde{W}$ is a standard Brownian motion martingale under P^* . It is well known that the form (4.33) is that of an L^2 -martingale with respect to $\{\mathcal{F}_t\}$, under P^* (cf. [11], Theorem 6.2). Thus, P^* is an equivalent martingale measure. In fact, P^* is the unique equivalent martingale measure.

An *admissible strategy* is a self-financing trading strategy ϕ such that $\{V_t^*(\phi), \mathcal{F}_t, t \in [0, T]\}$ is a martingale under the equivalent martingale measure P^* .

Given the stochastic differential equation (4.34) satisfied by S^* under P^* and the form (4.14) of $V^*(\phi)$, a sufficient condition for a self-financing trading strategy ϕ to be admissible is that

$$E^{P^*} \left[\int_0^T |\alpha_s S_s^*|^2 ds \right] < \infty. \quad (4.35)$$

In this case, the discounted value process will be an L^2 -martingale under P^* (cf. Appendix D). The condition (4.35) holds if α is a bounded process.

4.4. European Contingent Claims

A *European contingent claim* is represented by an $\mathcal{F}_T$ -measurable random variable X . A *replicating (or hedging) strategy* for a European contingent claim $X \in \mathcal{F}_T$ is an admissible strategy ϕ such that $V_T(\phi) = X$.

We shall use the following form of the *martingale representation theorem* for Brownian motion to obtain a replicating strategy for suitably integrable X . A more general form of this result is stated in Appendix D. A simpler version is stated here for convenience. For this theorem, it is important that a suitable filtration generated by a Brownian motion under the ambient probability measure is used. For this, note that $\tilde{W}$ is a standard one-dimensional Brownian motion under P^* . Now, since W and $\tilde{W}$ differ only by a deterministic process, it follows that $\sigma\{W_s : 0 \leq s \leq t\}$ is the same as $\sigma\{\tilde{W}_s : 0 \leq s \leq t\}$ for each $t \in [0, T]$. Furthermore, since P and P^* have the same null sets, augmenting by the P^* -null sets has the same effect as augmenting by the P -null sets. Consequently, the filtration $\{\mathcal{F}_t\}$ that was originally defined using W and augmentation by the P -null sets also satisfies $\mathcal{F}_t = \sigma\{\tilde{W}_s, s \in [0, t]\}^\sim$ for each $t \in [0, T]$, where the superscript $\sim$ denotes augmentation by the P^* -null sets. Consequently, $\{\mathcal{F}_t, t \in [0, T]\}$ is also the standard filtration generated by the Brownian motion $\tilde{W}$ on $(\Omega, \mathcal{F}, P^*)$. For a proof of the following martingale representation theorem, see Revuz and Yor [35], Theorem V.3.4.

Theorem 4.4.1. (*Martingale Representation Theorem*) Suppose that $M = \{M_t, \mathcal{F}_t, t \in [0, T]\}$ is a right continuous martingale under P^* . Then there exists a $(\mathcal{B}_T \times \mathcal{F}_T)$ -measurable, adapted process $\eta = \{\eta_t, t \in [0, T]\}$ such that P^* -a.s., $\int_0^T \eta_s^2 ds < \infty$ and

$$M_t = M_0 + \int_0^t \eta_s d\tilde{W}_s \quad \text{for all } t \in [0, T]. \quad (4.36)$$

Theorem 4.4.2. Suppose that X is an $\mathcal{F}_T$ -measurable random variable such that $E^{P^*}[|X|] < \infty$. Then there exists a replicating strategy ϕ for X . Moreover, for any replicating strategy ϕ and $X^* = X/B_T$, we have that for each $t \in [0, T]$, P^* -a.s.,

$$V_t^*(\phi) = E^{P^*}[X^* | \mathcal{F}_t]. \quad (4.37)$$

Proof. We first prove that (4.37) holds if ϕ is a replicating strategy for X . Indeed, for such a ϕ , we have for fixed $t \in [0, T]$ that P^* -a.s.,

$$V_t^*(\phi) = E^{P^*}[V_T^*(\phi) | \mathcal{F}_t] = E^{P^*}[X^* | \mathcal{F}_t], \quad (4.38)$$

where the first equality follows from the admissibility of ϕ , which implies that $\{V_t^*(\phi), \mathcal{F}_t, t \in [0, T]\}$ is a martingale under P^* , and the second equality follows from the replicating property of ϕ : $V_T(\phi) = X$.

In light of the above, to prove the existence of a replicating strategy ϕ , it is natural to consider

$$M_t = E^{P^*}[X^* | \mathcal{F}_t], \quad t \in [0, T]. \quad (4.39)$$

Note that $\{M_t, \mathcal{F}_t, t \in [0, T]\}$ is a P^* -martingale and hence it has a right continuous modification. In fact, since the filtration is generated by Brownian motion, M has a continuous modification (cf. Revuz and Yor [35], Theorem V.3.5). We again denote such a continuous modification by M . It follows from the martingale representation theorem, Theorem 4.4.1, that there is an adapted process $\eta = \{\eta_t, t \in [0, T]\}$ satisfying the conditions in Theorem 4.4.1 and such that P^* -a.s. for each $t \in [0, T]$,

$$M_t = M_0 + \int_0^t \eta_s d\tilde{W}_s. \quad (4.40)$$

Define

$$\alpha_t = \frac{\eta_t}{\sigma S_t^*}, \quad t \in [0, T]. \quad (4.41)$$

Then $\alpha = \{\alpha_t, t \in [0, T]\}$ is $(\mathcal{B}_T \times \mathcal{F}_T)$ -measurable, adapted and

$$\int_0^T \alpha_t^2 dt \leq \left(\sigma \inf_{t \in [0, T]} S_t^* \right)^{-2} \int_0^T \eta_t^2 dt. \quad (4.42)$$

Note that $S_t^* > 0$ for each $t \in [0, T]$, and S_t^* is a continuous function of $t \in [0, T]$. Therefore, $\inf_{t \in [0, T]} S_t^* > 0$. So it follows from the integrability property of η that P^* -a.s.,

$$\int_0^T \alpha_t^2 dt < \infty. \quad (4.43)$$

Define

$$\beta_t = M_t - \alpha_t S_t^*, \quad t \in [0, T]. \quad (4.44)$$

Then $\beta = \{\beta_t, t \in [0, T]\}$ is $(\mathcal{B}_T \times \mathcal{F}_T)$ -measurable and adapted, and using the Cauchy-Schwarz inequality we have

$$\begin{aligned} \int_0^T |\beta_t| dt &\leq \int_0^T |M_t| dt + \int_0^T |\alpha_t| S_t^* dt \\ &\leq T \max_{t \in [0, T]} |M_t| + \left(\max_{t \in [0, T]} S_t^* \right) \left(\int_0^T \alpha_t^2 dt \right)^{\frac{1}{2}} T^{\frac{1}{2}}. \end{aligned}$$

The last line above is finite P^* -a.s. by (4.43) and the fact that M and S^* are continuous processes. Let $\phi = \{(\alpha_t, \beta_t), t \in [0, T]\}$. Then,

$$V_t^*(\phi) = \alpha_t S_t^* + \beta_t = M_t, \quad t \in [0, T]. \quad (4.45)$$

This, together with (4.36), the definition of α , and (4.34), gives P^* -a.s.,

$$\begin{aligned} V_t^*(\phi) &= V_0(\phi) + \int_0^t \eta_s d\tilde{W}_s \\ &= V_0(\phi) + \int_0^t \alpha_s \sigma S_s^* d\tilde{W}_s \\ &= V_0(\phi) + \int_0^t \alpha_s dS_s^*, \end{aligned}$$

for each $t \in [0, T]$. Therefore, since P and P^* have the same null sets and $V^*(\phi) = M$ is continuous and adapted, ϕ is a self-financing trading strategy by Lemma 4.2.1. Combining this with (4.45) and the martingale property of M , we see that ϕ is admissible. Finally, taking $t = T$ in (4.45) implies that ϕ replicates X since $M_T = X^*$. $\square$

4.5. Pricing European Contingent Claims

Consider a European contingent claim $X \in \mathcal{F}_T$ such that $E^{P^*}[|X|] < \infty$. To determine the arbitrage free price process for this claim, we need a notion of admissible strategy for trading in stock, bond and the contingent claim.

We assume that any given price process $C = \{C_t, t \in [0, T]\}$ for the European contingent claim is a right continuous, adapted process, where C_t represents the price of the European contingent claim at time t . (Adaptedness

is the minimal assumption we would expect on this process, and right continuity is a reasonable regularity assumption.) We suppose that for a given price process C , there is a set of admissible strategies Ψ_C for trading in stock, bond and the contingent claim. At a minimum, an element ψ of Ψ_C should be a three-dimensional adapted process $\psi = \{\psi_t = (\alpha_t, \beta_t, \gamma_t), t \in [0, T]\}$. We shall also want elements of Ψ_C to be self-financing in an appropriate way.

To show uniqueness of the price process, we do not want to place unnecessary restrictions on C just so that it can be used as an integrator (in defining self-financing trading strategies). In fact, to show uniqueness, we shall need only the reasonable assumption that rather simple strategies of the following type are admissible: those where one buys or sells one contingent claim, or does nothing, at some deterministic time t^* and then holds that position in the claim until time T , but where one may trade as usual in the stock and bond over $[t^*, T]$. More precisely, we assume that strategies of the following form are in Ψ_C :

$$\psi_t(\omega) = 1_A(\omega)(\tilde{\alpha}_t(\omega), \tilde{\beta}_t(\omega) + \tilde{\delta}_t(\omega), \tilde{\gamma}_t(\omega)), \quad \text{for } t \in [t^*, T], \quad (4.46)$$

and $\psi_t(\omega) = 0$ for $t \in [0, t^*)$, for all $\omega \in \Omega$, where t^* is a fixed time in $[0, T]$, $A \in \mathcal{F}_{t^*}$, $\{(\tilde{\alpha}_t, \tilde{\beta}_t), t \in [0, T]\}$ is an admissible self-financing trading strategy in the primary (stock-bond) market, $\tilde{\gamma}_t = +1$ for all $t \in [t^*, T]$ or $\tilde{\gamma}_t = -1$ for all $t \in [t^*, T]$, $\tilde{\delta}_t = \tilde{\delta}_{t^*}$ for all $t \in [t^*, T]$ is chosen such that $\tilde{\alpha}_{t^*}S_{t^*} + (\tilde{\beta}_{t^*} + \tilde{\delta}_{t^*})B_{t^*} + \tilde{\gamma}_{t^*}C_{t^*} = 0$. Under such a strategy, the investor does nothing prior to time t^* . On A^c , the investor continues to do nothing for the remaining interval $[t^*, T]$. On A , if $\tilde{\gamma}_{t^*} = +1$, at t^* the investor buys one contingent claim for C_{t^*} and invests $-C_{t^*}$ in the stock-bond market according to $(\tilde{\alpha}_t, \tilde{\beta}_t + \tilde{\delta}_t)$ for $t \in [t^*, T]$. That is, on A , for $t \in [t^*, T]$, the investor follows the strategy $(\tilde{\alpha}, \tilde{\beta})$ in the primary market and, in addition, purchases $\tilde{\delta}_{t^*}$ units of the bond at time t^* and holds those until time T . On A , if $\tilde{\gamma}_{t^*} = -1$, instead of buying one contingent claim at time t^* , the investor sells one contingent claim at t^* and invests the proceeds C_{t^*} according to $(\tilde{\alpha}_t, \tilde{\beta}_t + \tilde{\delta}_t)$ for $t \in [t^*, T]$. Since the value of the strategy ψ is zero at time t^* and $\tilde{\delta}_t$ and $\tilde{\gamma}_t$ do not vary with $t \geq t^*$, the strategy will be naturally self-financing, since $(\tilde{\alpha}, \tilde{\beta})$ is assumed to have this property.

To show existence of the price process, we will prove below that when the price process $C = \{C_t, t \in [0, T]\}$ is a continuous modification of

$$e^{rt}E^{P^*}[X^* | \mathcal{F}_t], \quad t \in [0, T], \quad (4.47)$$

it is an arbitrage free price process. For this, we need constraints on the admissible strategies similar to what we had in the case of the primary market to rule out doubling type strategies. Note that in this case, if $\phi^* = (\alpha^*, \beta^*)$ is a replicating strategy for X , then $C_t^* = e^{-rt}C_t = V_t^*(\phi^*)$ for

$t \in [0, T]$ defines a continuous martingale under P^* and P^* -a.s.,

$$dC_t^* = dV_t^*(\phi^*) = \alpha_t^* dS_t^* = \sigma \alpha_t^* S_t^* d\tilde{W}_t. \quad (4.48)$$

Consequently, in this case, we let the class of admissible strategies Ψ_C be the set of three-dimensional processes $\psi = (\alpha, \beta, \gamma)$ such that

- (i) $\psi : [0, T] \times \Omega \rightarrow \mathbb{R}^3$ is $(\mathcal{B}_T \times \mathcal{F}_T)$ -measurable, where $\psi(t, \omega) = \psi_t(\omega)$ for $t \in [0, T]$ and $\omega \in \Omega$;
- (ii) ψ is adapted, i.e., $\psi_t \in \mathcal{F}_t$ for each $t \in [0, T]$;
- (iii) $\int_0^T \alpha_t^2 dt < \infty$, $\int_0^T |\beta_t| dt < \infty$, $\int_0^T \gamma_t^2 (\alpha_t^*)^2 dt < \infty$, P^* -a.s.;
- (iv) the value process $\{V_t(\psi), t \in [0, T]\}$ defined by

$$V_t(\psi) = \alpha_t S_t + \beta_t B_t + \gamma_t C_t, \quad t \in [0, T], \quad (4.49)$$

is a continuous, adapted process such that the discounted value process $\{V_t^*(\psi), t \in [0, T]\}$ defined by

$$V_t^*(\psi) = V_t(\psi)/B_t, \quad t \in [0, T], \quad (4.50)$$

satisfies P^* -a.s. for each $t \in [0, T]$:

$$\begin{aligned} V_t^*(\psi) &= V_0(\psi) + \int_0^t \alpha_s dS_s^* + \int_0^t \gamma_s dC_s^*, \\ &= V_0(\psi) + \sigma \int_0^t \alpha_s S_s^* d\tilde{W}_s + \sigma \int_0^t \gamma_s \alpha_s^* S_s^* d\tilde{W}_s; \end{aligned} \quad (4.51)$$

- (v) $\{V_t^*(\psi), \mathcal{F}_t, t \in [0, T]\}$ is a martingale under P^* .

Properties (i)–(iii) above ensure that the integrals appearing in (4.51) are well defined, property (iv) is the self-financing property of ψ (cf. Lemma 4.2.1), and (v) amounts to a constraint on the value of ψ to rule out doubling strategies. Note that since S^* and C^* are already martingales under P^* , this last condition amounts to some control on the size of ψ . One can verify that strategies of the form (4.46) satisfy the above properties (i)–(v) when C is a continuous modification of (4.47).

Given a price process C for the contingent claim, an *arbitrage opportunity* in the stock-bond-contingent claim market is an admissible strategy $\psi \in \Psi_C$ such that $V_0(\psi) = 0$, $V_T(\psi) \geq 0$ and $P^*(V_T(\psi) > 0) > 0$. (Since P and P^* are equivalent, we could just as easily have used P in place of P^* in the last inequality.)

In the following theorem, uniqueness means uniqueness up to indistinguishability of stochastic processes.

Theorem 4.5.1. *The unique arbitrage free price process for the P^* -integrable European contingent claim X is a continuous modification of*

$$\{e^{rt} E^{P^*} [X^* | \mathcal{F}_t], \quad t \in [0, T]\}.$$

Proof. Consider a price process $\{C_t, t \in [0, T]\}$ for the European contingent claim. By Theorem 4.4.2, there is a replicating strategy $\phi^* = (\alpha^*, \beta^*)$ for X , and its value process $\{V_t(\phi^*), t \in [0, T]\}$ is a continuous modification of $\{e^{rt} E^{P^*}[X^* | \mathcal{F}_t], t \in [0, T]\}$. Suppose that

$$P^*(C_t \neq V_t(\phi^*) \text{ for some } t \in [0, T]) > 0.$$

Since the process C is assumed to be right continuous and $V(\phi^*)$ is continuous, there exists a $t^* \in [0, T]$ such that $P^*(C_{t^*} \neq V_{t^*}(\phi^*)) > 0$. Let $A = \{\omega \in \Omega : C_{t^*}(\omega) > V_{t^*}(\phi^*)(\omega)\}$ and $\tilde{A} = \{\omega \in \Omega : C_{t^*}(\omega) < V_{t^*}(\phi^*)(\omega)\}$. Then either $P^*(A) > 0$ or $P^*(\tilde{A}) > 0$. First suppose that $P^*(A) > 0$. Define $\psi = \{(\alpha_t, \beta_t, \gamma_t), t \in [0, T]\}$ by

$$\psi_t(\omega) = \begin{cases} (0, 0, 0), & t < t^*, \\ (0, 0, 0), & t \geq t^*, \omega \in A^c, \\ \left(\alpha_t^*(\omega), \beta_t^*(\omega) + \frac{C_{t^*}(\omega) - V_{t^*}(\phi^*)(\omega)}{B_{t^*}}, -1\right), & t \geq t^*, \omega \in A, \end{cases}$$

or in other words, for $t \in [0, T]$, $\omega \in \Omega$,

$$\psi_t(\omega) = 1_{\{t \geq t^*\}} 1_A(\omega) \left(\alpha_t^*, \beta_t^* + \frac{C_{t^*} - V_{t^*}(\phi^*)}{B_{t^*}}, -1 \right) (\omega). \quad (4.52)$$

Clearly, the value of the portfolio represented by ψ_t is zero for $t \leq t^*$, and so the value at time zero is zero. Moreover, the value of ψ at time T is zero on A^c . Also, since ϕ^* is a replicating strategy for X , on A , the value at time T is $\left(\frac{C_{t^*} - V_{t^*}(\phi^*)}{B_{t^*}}\right) B_T$, which is strictly positive. Since $P^*(A) > 0$, ψ is an arbitrage opportunity.

Similarly, if $P^*(\tilde{A}) > 0$, then ψ defined for $t \in [0, T]$, $\omega \in \Omega$ by

$$\psi_t(\omega) = 1_{\{t \geq t^*\}} 1_{\tilde{A}}(\omega) \left(-\alpha_t^*, -\beta_t^* + \frac{V_{t^*}(\phi^*) - C_{t^*}}{B_{t^*}}, 1 \right) (\omega), \quad (4.53)$$

is an arbitrage opportunity.

Thus, we have shown that the only possible arbitrage free price process (up to indistinguishability) is given by $C_t = V_t(\phi^*)$, $t \in [0, T]$. Next we show that this price process is arbitrage free. For a contradiction, suppose that with this price process, there is $\psi \in \Psi_C$ such that $V_0(\psi) = 0$, $V_T(\psi) \geq 0$ and $E^{P^*}[V_T(\psi)] > 0$. Now, by the admissibility assumption on ψ , the discounted value process $V^*(\psi)$ is a martingale under P^* and so

$$0 = V_0^*(\psi) = E^{P^*}[V_T^*(\psi)] = e^{-rT} E^{P^*}[V_T(\psi)] > 0, \quad (4.54)$$

a contradiction. Thus, there cannot be an arbitrage opportunity with this price process. $\square$

4.6. European Call Option — Black-Scholes Formula

In this section, we apply the results of the previous section to compute the arbitrage free price process for a European call option, and we also identify a replicating strategy for the option.

Let $X = (S_T - K)^+$, which represents the value of a *European call option* with strike price $K \in (0, \infty)$ and expiration time T . Notice that $0 \leq X \leq S_T$ and $E^{P^*}[S_T] = S_0 e^{rT} < \infty$.

4.6.1. Price Process. By Theorem 4.5.1, the arbitrage free price process for the European call option X is given by $\{C_t, t \in [0, T]\}$, which is a continuous adapted process such that for each $t \in [0, T]$, if $C_t^* = C_t e^{-rt}$, then we have P^* -a.s.,

$$C_t^* = E^{P^*}[X^* | \mathcal{F}_t] = E^{P^*}[(S_T^* - K^*)^+ | \mathcal{F}_t], \quad (4.55)$$

where $K^* = e^{-rT}K$. It follows from (4.33) that P^* -a.s., for each $t \in [0, T]$,

$$\begin{aligned} S_T^* &= S_t^* \frac{S_0 \exp\left(\sigma \tilde{W}_T - \frac{1}{2}\sigma^2 T\right)}{S_0 \exp\left(\sigma \tilde{W}_t - \frac{1}{2}\sigma^2 t\right)} \\ &= S_t^* \exp\left(\sigma (\tilde{W}_T - \tilde{W}_t) - \frac{1}{2}\sigma^2(T-t)\right). \end{aligned}$$

On substituting this into the right member of (4.55), we obtain that for $t \in [0, T]$, P^* -a.s.,

$$C_t^* = E^{P^*}\left[\left(S_t^* \exp\left(\sigma (\tilde{W}_T - \tilde{W}_t) - \frac{1}{2}\sigma^2(T-t)\right) - K^*\right)^+ \middle| \mathcal{F}_t\right]. \quad (4.56)$$

Using the fact that $S_t^* \in \mathcal{F}_t$ and that $\tilde{W}_T - \tilde{W}_t$ is a normal random variable with mean zero and variance $T-t$ that is independent of $\mathcal{F}_t$ under P^* (since $\mathcal{F}_t$ is generated by $\tilde{W}_{\cdot \wedge t}$ and the P^* -null sets), it follows that for $t \in [0, T]$, P^* -a.s.,

$$C_t^* = \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} \left(S_t^* e^{\sigma y - \frac{1}{2}\sigma^2(T-t)} - K^*\right)^+ e^{-\frac{y^2}{2(T-t)}} dy,$$

and $C_T^* = (S_T^* - K^*)^+$. Let $z = y/\sqrt{T-t}$. Then the expression on the right above becomes

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(S_t^* e^{\sigma z \sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)} - K^*\right)^+ e^{-\frac{1}{2}z^2} dz. \quad (4.57)$$

Define $f : [0, T] \times (0, \infty) \rightarrow [0, \infty)$ by $f(T, x) = (x - K^*)^+$ for all $x \in (0, \infty)$, and for $t \in [0, T)$, $x \in (0, \infty)$,

$$f(t, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(x e^{\sigma z \sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)} - K^*\right)^+ e^{-\frac{1}{2}z^2} dz. \quad (4.58)$$

Then, by the above, for each $t \in [0, T]$, P^* -a.s.,

$$C_t^* = f(t, S_t^*). \quad (4.59)$$

For $t \in [0, T)$ and $x \in (0, \infty)$, let $l_t(x)$ be such that the integrand in (4.58) is strictly positive if and only if $z > l_t(x)$; i.e.,

$$l_t(x) = \frac{\log\left(\frac{K^*}{x}\right)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}. \quad (4.60)$$

Let Φ denote the cumulative distribution function for a standard normal random variable with mean zero and variance one:

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\frac{1}{2}u^2} du, \quad z \in (-\infty, \infty). \quad (4.61)$$

Then, for $t \in [0, T)$ and $x \in (0, \infty)$,

$$\begin{aligned} f(t, x) &= \frac{x}{\sqrt{2\pi}} \int_{l_t(x)}^{\infty} e^{\sigma z\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t) - \frac{1}{2}z^2} dz - \frac{K^*}{\sqrt{2\pi}} \int_{l_t(x)}^{\infty} e^{-\frac{1}{2}z^2} dz \\ &= \frac{x}{\sqrt{2\pi}} \int_{l_t(x)}^{\infty} e^{-\frac{1}{2}(z - \sigma\sqrt{T-t})^2} dz - K^* \Phi(-l_t(x)) \\ &= \frac{x}{\sqrt{2\pi}} \int_{l_t(x) - \sigma\sqrt{T-t}}^{\infty} e^{-\frac{1}{2}u^2} du - K^* \Phi(-l_t(x)) \\ &= x\Phi\left(-l_t(x) + \sigma\sqrt{T-t}\right) - K^* \Phi(-l_t(x)). \end{aligned}$$

Here we have used the change of variable $u = z - \sigma\sqrt{T-t}$ and the symmetry of Φ . Thus, for $t \in [0, T)$, P^* -a.s.,

$$\begin{aligned} C_t^* &= f(t, S_t^*) \\ &= S_t^* \Phi\left(\frac{\log\left(\frac{S_t^*}{K^*}\right)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}\right) \\ &\quad - K^* \Phi\left(\frac{\log\left(\frac{S_t^*}{K^*}\right)}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t}\right), \end{aligned}$$

and in particular, P^* -a.s.,

$$\begin{aligned} C_0 &= C_0^* \\ &= S_0 \Phi\left(\frac{\log\left(\frac{S_0}{K^*}\right)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}\right) \\ &\quad - K^* \Phi\left(\frac{\log\left(\frac{S_0}{K^*}\right)}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T}\right), \end{aligned} \quad (4.62)$$

which is the famous *Black-Scholes formula*.

4.6.2. Implied Volatility. Note that the Black-Scholes formula (4.62) does not involve the rate of return μ of the stock. The only stock price parameter that appears in the formula is the volatility σ . Given this volatility and assuming the Black-Scholes model for the stock price, one can use the Black-Scholes formula to price a European call option with a given strike price K and expiration time T . For this application, one might hope to use historical data to estimate the volatility parameter. However, such estimates typically produce biased option prices. Another method is often used to estimate volatility; namely, the Black-Scholes formula is used in reverse to obtain an estimate of the volatility of the underlying stock price process from market prices of frequently traded options; e.g., those traded on an exchange. (This is generally a meaningful procedure, as the initial arbitrage free price C_0 given by (4.62) can be verified by differentiation to be a strictly increasing function of the volatility σ for fixed K and T .) An estimate of volatility obtained in this way is called *implied volatility*. This can then be used as an input to the Black-Scholes model to determine prices of other European contingent claims that are not traded frequently.

All European call options based on a given stock use the same volatility parameter σ for their pricing, regardless of the strike price or the expiration date of the option. This would yield a horizontal line for the graph of volatility as a function of strike price. On the other hand, empirical plots of implied volatility as a function of strike price often have a U-shape or half U-shape, referred to as a *smile* or *smirk*. Various efforts have been made to generalize the Black-Scholes model to overcome this deficiency. One approach involves generalizing the model to allow the constant volatility σ to be replaced by a continuous function of a one-dimensional Itô process Z where the driving Brownian motion W' for Z is one-dimensional and W' may be correlated with the Brownian motion W driving the stock price process S . Such models are called stochastic volatility models. They are generally *incomplete*, in that not all suitably integrable contingent claims can be replicated. For a discussion of such models, see Chapter 6 of Musiela and Rutkowski [32], and for an in-depth treatment, see the monograph by Fouque, Papanicolaou and Sircar [17]. An example of a stochastic volatility model is given in the Exercises for Chapter 5.

4.6.3. Replicating Strategy. The Black-Scholes formula gives the arbitrage free initial price for the European call option, but it does not immediately give a replicating or hedging strategy for the option. Recall that the existence of such a hedging strategy was guaranteed by the martingale representation theorem (cf. the proof of Theorem 4.4.2), but that result is non-constructive. We will now use stochastic calculus and the function f defined above to obtain such a hedging strategy.

Note that $f : [0, T] \times (0, \infty) \rightarrow [0, \infty)$ is defined for $t \in [0, T]$, $x \in (0, \infty)$ by

$$\begin{aligned} f(t, x) = & x\Phi\left(\frac{\log\left(\frac{x}{K^*}\right)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}\right) \\ & - K^*\Phi\left(\frac{\log\left(\frac{x}{K^*}\right)}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t}\right) \end{aligned}$$

and

$$f(T, x) = (x - K^*)^+. \quad (4.63)$$

It is straightforward to verify from the above formula that $f \in C^{1,2}([0, T] \times (0, \infty))$; i.e., as a function of $(t, x) \in [0, T] \times (0, \infty)$, $f(t, x)$ is once continuously differentiable in t and twice continuously differentiable in x . In particular, the first partial derivative with respect to x is given for $t \in [0, T]$, $x \in (0, \infty)$, by

$$f_x(t, x) = \Phi\left(\frac{\log\left(\frac{x}{K^*}\right)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}\right), \quad (4.64)$$

which is a bounded continuous function on $[0, T] \times (0, \infty)$. In addition, by using the alternative representation (cf. (4.56)):

$$f(t, x) = E^{P^*} \left[\left(x \exp \left(\sigma(\tilde{W}_T - \tilde{W}_t) - \frac{1}{2}\sigma^2(T-t) \right) - K^* \right)^+ \right]$$

for $(t, x) \in [0, T] \times (0, \infty)$, the uniform integrability of $\{\exp(\sigma(\tilde{W}_T - \tilde{W}_t) - \frac{1}{2}\sigma^2(T-t)), t \in [0, T]\}$, and dominated convergence, we can readily verify that f is continuous on $[0, T] \times (0, \infty)$. Note that by the continuity of f , S_t^* , and C_t^* , we may assume that P^* -a.s., $C_t^* = f(t, S_t^*)$ for all $t \in [0, T]$, (i.e., the exceptional P^* -null set does not depend on t). We can apply Itô's formula to $f(t, S_t^*)$ for $t < T$ to obtain P^* -a.s., for all $t < T$,

$$\begin{aligned} f(t, S_t^*) = & f(0, S_0^*) + \int_0^t f_s(s, S_s^*) ds + \int_0^t f_x(s, S_s^*) dS_s^* \\ & + \frac{1}{2} \int_0^t f_{xx}(s, S_s^*) d[S^*]_s, \end{aligned} \quad (4.65)$$

where $f_s(s, x)$ denotes the first partial derivative of $f(s, x)$ with respect to s ; $f_x(s, x)$ and $f_{xx}(s, x)$ denote the first and second partial derivatives, respectively, of $f(s, x)$ with respect to x ; and $[S^*]$ denotes the quadratic variation process associated with S^* which satisfies $d[S^*]_s = \sigma^2(S_s^*)^2 ds$ (cf. (4.34)). The first and third integrals above are path-by-path Riemann integrals, and the second integral is a stochastic integral. Indeed, for each $t < T$, $\{\int_0^u f_x(s, S_s^*) dS_s^*, \mathcal{F}_u, u \in [0, t]\}$ is a continuous local martingale

starting from zero under P^* . On the other hand, $\{C_u^*, \mathcal{F}_u, u \in [0, t]\}$ is a continuous process and by (4.55) it is a martingale under P^* . Now, P^* -a.s., $C_u^* = f(u, S_u^*)$ for all $u \in [0, t]$, and so by (4.65) we have P^* -a.s. for all $u \in [0, t]$:

$$\begin{aligned} & \int_0^u \left(f_s(s, S_s^*) + \frac{1}{2} \sigma^2 (S_s^*)^2 f_{xx}(s, S_s^*) \right) ds \\ &= C_u^* - C_0^* - \int_0^u f_x(s, S_s^*) dS_s^*. \end{aligned} \quad (4.66)$$

Now, by the above discussion, the right side of the equals sign above defines a continuous local martingale under P^* that starts from zero, and the left side defines a continuous, adapted process whose paths are of bounded variation on $[0, t]$. It follows (cf. [11], Corollary 4.5), that P^* -a.s., both sides of (4.66) are zero for all $u \in [0, t]$. Since $t < T$ was arbitrary, this in fact holds true for all $u \in [0, T)$. Indeed, not only is the left side zero P^* -a.s., but one can show (for example by direct verification using the formula for f) that f satisfies the *Black-Scholes partial differential equation*:

$$f_t + \frac{1}{2} \sigma^2 x^2 f_{xx} = 0 \quad \text{for all } (t, x) \in [0, T) \times (0, \infty).$$

Remark. There are several different forms of the Black-Scholes partial differential equation, depending on whether one uses the function $f(t, x)$ for representing C^* or one uses the undiscounted function $e^{rt} f(t, x)$ for representing C .

Using the fact that P^* -a.s., the right side of (4.66) is zero for all $u \in [0, T)$, we have that P^* -a.s.,

$$C_t^* = C_0 + \int_0^t f_x(s, S_s^*) dS_s^* \quad \text{for all } t < T. \quad (4.67)$$

(Note that $C_0 = C_0^*$ for this.) Comparing this with the self-financing property of trading strategies and keeping in mind that we want C^* to be the discounted value process for a replicating strategy, we define

$$\alpha_t = f_x(t, S_t^*) \quad \text{for each } t \in [0, T).$$

The definition of α_T is not critical (since it is only in force for an instant of time) as long as it is part of a self-financing strategy. However, we can define it in such a way that $\alpha_T = \lim_{t \rightarrow T} \alpha_t$ P^* -a.s. Note that since $P^*(S_T^* = K^*) = 0$ and S^* has continuous paths, we have that P^* -a.s.,

$$\begin{aligned} \lim_{t \rightarrow T} f_x(t, S_t^*) &= \lim_{t \rightarrow T} \Phi \left(\frac{\log \left(\frac{S_t^*}{K^*} \right)}{\sigma \sqrt{T-t}} + \frac{1}{2} \sigma \sqrt{T-t} \right) \\ &= 1_{\{S_T^* > K^*\}}. \end{aligned} \quad (4.68)$$

Thus, we may and do define $\alpha_T = 1_{\{S_T^* > K^*\}}$. In order to maintain the appropriate value process, we define

$$\beta_t = C_t^* - \alpha_t S_t^*, \quad t \in [0, T].$$

It is straightforward to verify that $\phi = \{(\alpha_t, \beta_t), t \in [0, T]\}$ is a trading strategy with discounted value process C^* . By the continuity of C^* and of the stochastic integral process $\{\int_0^t \alpha_s dS_s^*, t \in [0, T]\}$, it follows on taking the limit as $t \rightarrow T$ in (4.67) that we have P^* -a.s., for each $t \in [0, T]$,

$$V_t^*(\phi) = \alpha_t S_t^* + \beta_t = C_t^* = C_0 + \int_0^t \alpha_s dS_s^*,$$

which implies that ϕ is self-financing. Moreover, since $\{C_t^*, \mathcal{F}_t, t \in [0, T]\}$ is a martingale under P^* , ϕ is an admissible strategy. Finally, since $V_T(\phi) = C_T = (S_T - K)^+$, ϕ is a replicating strategy for the European call option with strike price K and expiration time T . The results of this section are summarized in the following theorem.

Theorem 4.6.1. *For each $t \in [0, T)$, let*

$$\begin{aligned} C_t^* &= S_t^* \Phi \left(\frac{\log \left(\frac{S_t^*}{K^*} \right)}{\sigma \sqrt{T-t}} + \frac{1}{2} \sigma \sqrt{T-t} \right) \\ &\quad - K^* \Phi \left(\frac{\log \left(\frac{S_t^*}{K^*} \right)}{\sigma \sqrt{T-t}} - \frac{1}{2} \sigma \sqrt{T-t} \right), \end{aligned} \quad (4.69)$$

and let $C_T^* = (S_T^* - K^*)^+$. Define $\alpha_T = 1_{\{S_T^* > K^*\}}$, and let

$$\begin{aligned} \alpha_t &= \Phi \left(\frac{\log \left(\frac{S_t^*}{K^*} \right)}{\sigma \sqrt{T-t}} + \frac{1}{2} \sigma \sqrt{T-t} \right), \quad \text{for each } t \in [0, T), \\ \beta_t &= C_t^* - \alpha_t S_t^* \quad \text{for each } t \in [0, T]. \end{aligned}$$

Then $\phi = \{(\alpha_t, \beta_t), t \in [0, T]\}$ is a replicating strategy for the European call option with strike price K and expiration time T , and the arbitrage free price process for this option is given by $\{e^{rt} C_t^*, t \in [0, T]\}$.

4.7. American Contingent Claims

An *American contingent claim* (ACC) is represented by a non-negative, continuous, adapted stochastic process $Y = \{Y_t, t \in [0, T]\}$ evolving in continuous time such that $E^{P^*} [\sup_{0 \leq t \leq T} Y_t] < \infty$. The random variable Y_t , $t \in [0, T]$, is interpreted as the payoff for the claim if the owner cashes it in at time t . The time at which the owner cashes in the claim is required to be a stopping time taking values in $[0, T]$; i.e., it is a random variable

$\tau : \Omega \rightarrow [0, T]$ such that $\{\tau \leq t\} \in \mathcal{F}_t$ for each $t \in [0, T]$. For $0 \leq s \leq t \leq T$, let $\mathcal{T}_{[s, t]}$ denote the set of stopping times that take values in the interval $[s, t]$. An example of an American contingent claim is an *American call option* with strike price $K \in (0, \infty)$, which has payoff $Y_t = (S_t - K)^+$ at time t , $t \in [0, T]$. Note that if $S_t \leq K$, cashing in the contingent claim at time t is equivalent to not exercising the option at all. We have adopted this convention so that we can use one framework for treating all contingent claims, including options and contracts.

As noted in Section 2.3, devoted to pricing American contingent claims in the setting of the discrete binomial model, an important feature of an American contingent claim is that the buyer and the seller of such a derivative have different actions available to them: the buyer may cash in the claim at any stopping time $\tau \in \mathcal{T}_{[0, T]}$, whereas the seller seeks protection from the risk associated with all possible choices of the stopping time τ by the buyer. Also, as for the binomial model, in pricing and hedging American contingent claims, a critical role is played by a superhedging strategy that hedges the risk for the seller of an American contingent claim.

A *superhedging strategy* for the seller of an American contingent claim, $Y = \{Y_t, t \in [0, T]\}$, is an admissible strategy $\phi = \{(\alpha_t, \beta_t), t \in [0, T]\}$ with value $V_t(\phi)$ at time $t \in [0, T]$, such that $V_t(\phi) \geq Y_t$ for each $t \in [0, T]$, P^* -a.s. In the context of the binomial model, such a ϕ can be constructed stepwise by proceeding backwards through the binary tree. In continuous time, such a stepwise construction is no longer available. Nevertheless, one can exploit the theory of Snell envelopes to show that a superhedging strategy exists.

For a non-empty collection of non-negative random variables $\{X_\gamma, \gamma \in \Gamma\}$ defined on a probability space, their *essential supremum* is a random variable X satisfying

- (i) $X \geq X_\gamma$ a.s. for each $\gamma \in \Gamma$, and
- (ii) if Z is a random variable such that $Z \geq X_\gamma$ a.s. for each $\gamma \in \Gamma$, then $Z \geq X$ a.s.

We write $X = \text{ess sup}_{\gamma \in \Gamma} X_\gamma$. (For the existence and properties of the essential supremum, see Karatzas and Shreve [28], Appendix A. The essential supremum is unique, up to a.s. equivalence.)

For each $t \in [0, T]$, let $Y_t^* = \frac{Y_t}{B_t} = e^{-rt} Y_t$. Define a process $U^* = \{U_t^*, t \in [0, T]\}$ as follows. For each $t \in [0, T]$, let

$$U_t^* = \text{ess sup}_{\tau \in \mathcal{T}_{[t, T]}} E^{P^*} [Y_\tau^* \mid \mathcal{F}_t], \quad (4.70)$$

where we take U_t^* to be $\mathcal{F}_t$ -measurable, without loss of generality. Also, for $t = 0$, since $\mathcal{F}_0$ is trivial, we may take $U_0^* = \sup_{\tau \in \mathcal{T}_{[0, T]}} E^{P^*} [Y_\tau^*]$, and for $t = T$, we may take $U_T^* = Y_T^*$. The right member in (4.70) is finite P^* -a.s.

since we assumed that $Y_s \geq 0$ for each $s \in [0, T]$ and $E^{P^*}[\sup_{0 \leq s \leq T} Y_s] < \infty$. It is straightforward to verify that $\{U_t^*, \mathcal{F}_t, t \in [0, T]\}$ is a supermartingale. In fact (cf. [28], Appendix D), there is a modification of this supermartingale whose paths are right continuous with finite left limits (abbreviated as r.c.l.l.). Henceforth, we shall assume that U^* is such a “nice” modification. Let $\tau^*(t) \equiv \inf\{s \geq t : U_s^* = Y_s^*\}$ for each $t \in [0, T]$ and define $\tau^* = \tau^*(0)$. Note that $U_{\tau^*}^* = Y_{\tau^*}^*$ P^* -a.s., since U^* has right continuous paths, Y^* has continuous paths, and $U_T^* = Y_T^*$ P^* -a.s. For later use, we define

$$U_t = e^{rt} U_t^*, \quad t \in [0, T]. \quad (4.71)$$

The following result is proved in Appendix D of Karatzas and Shreve [28].

Lemma 4.7.1.

- (i) $\{U_t^*, \mathcal{F}_t, t \in [0, T]\}$ is the smallest P^* -supermartingale that has r.c.l.l. paths and satisfies P^* -a.s. $U_t^* \geq Y_t^*$ for each $t \in [0, T]$;
- (ii) $\tau^*(t) \in \mathcal{T}_{[t, T]}$ and

$$U_t^* = E^{P^*} \left[Y_{\tau^*(t)}^* \mid \mathcal{F}_t \right], \quad t \in [0, T];$$

- (iii) $\{U_{t \wedge \tau^*}^*, \mathcal{F}_t, t \in [0, T]\}$ is a martingale under P^* .

By the Doob-Meyer decomposition theorem (cf. [28], Theorem D.13), we have the following decomposition for the uniformly integrable supermartingale U^* :

$$U_t^* = U_0^* + M_t - A_t, \quad t \in [0, T], \quad (4.72)$$

where $M = \{M_t, t \in [0, T]\}$ is a P^* -martingale with r.c.l.l. paths; $M_0 = 0$; and $A = \{A_t, t \in [0, T]\}$ is a continuous, non-decreasing, adapted process such that $A_0 = 0$, $E^{P^*}[A_t] < \infty$ for each $t \in [0, T]$. In fact, this decomposition is unique (up to indistinguishability of stochastic processes). By the martingale representation theorem (Theorem 4.4.1), there is a $(\mathcal{B}_T \times \mathcal{F}_T)$ -measurable, adapted process $\eta = \{\eta_t, t \in [0, T]\}$ such that P^* -a.s., $\int_0^T \eta_s^2 ds < \infty$ and

$$M_t = \int_0^t \eta_s d\tilde{W}_s, \quad \text{for all } t \in [0, T]. \quad (4.73)$$

In particular, P^* -a.s., the paths of M are continuous, and so without loss of generality we may assume that all of the paths of M are continuous.

Let

$$\phi_t = (\alpha_t, \beta_t) = \left(\frac{\eta_t}{\sigma S_t^*}, U_0^* + M_t - \frac{\eta_t}{\sigma} \right), \quad t \in [0, T]. \quad (4.74)$$

Then, ϕ is a trading strategy since it satisfies the measurability and local integrability properties required of a trading strategy (cf. Section 4.2). The

discounted value process for ϕ satisfies P^* -a.s. for each $t \in [0, T]$:

$$\begin{aligned}
 V_t^*(\phi) &= \frac{\eta_t}{\sigma} + U_0^* + M_t - \frac{\eta_t}{\sigma} \\
 &= U_0^* + M_t \\
 &= U_0^* + \int_0^t \eta_s d\tilde{W}_s \\
 &= U_0^* + \int_0^t \alpha_s \sigma S_s^* d\tilde{W}_s \\
 &= V_0(\phi) + \int_0^t \alpha_s dS_s^*, \tag{4.75}
 \end{aligned}$$

where the third line is by (4.73), the fourth is by (4.74), and the fifth is by (4.34). We also used the fact that $U_0^* = V_0^*(\phi) = V_0(\phi)$. It follows from Lemma 4.2.1 that ϕ is self-financing. Moreover, since U_0^* is a finite constant and M is a martingale under P^* , $\{V_t^*(\phi), \mathcal{F}_t, t \in [0, T]\}$ is a P^* -martingale. Hence, ϕ is an admissible strategy.

Next we verify that ϕ is a superhedging strategy. For this note that by (4.75), (4.72) and (i) of Lemma 4.7.1, we have P^* -a.s.,

$$V_t^*(\phi) = U_0^* + M_t = U_t^* + A_t \geq U_t^* \geq Y_t^*, \quad t \in [0, T]. \tag{4.76}$$

Finally, by Lemma 4.7.1, (4.72), and the uniqueness of the Doob-Meyer decomposition, P^* -a.s.,

$$U_{t \wedge \tau^*}^* = U_0^* + M_{t \wedge \tau^*} = V_{t \wedge \tau^*}^*(\phi), \quad t \in [0, T]. \tag{4.77}$$

We now argue that $U_0 = U_0^*$ is the unique arbitrage free initial price for the American contingent claim at time zero. For this, we need the notion of an arbitrage in a market where the stock and bond can be traded, and the American contingent claim (ACC) can be bought or sold at time zero. For such a market, let the initial price of the ACC be a constant C_0 . There are two types of arbitrage opportunities: one for a seller and another for a buyer of the ACC. The *seller* of the ACC has an *arbitrage opportunity* if there is an admissible strategy ϕ^s in stock and bond such that $V_0(\phi^s) = C_0$ and for all stopping times $\tau \in \mathcal{T}_{[0, T]}$,

$$V_\tau(\phi^s) - Y_\tau \geq 0 \quad \text{and} \quad E^{P^*} [V_\tau(\phi^s) - Y_\tau] > 0. \tag{4.78}$$

The *buyer* of the ACC has an *arbitrage opportunity* if there is an admissible strategy ϕ^b such that $V_0(\phi^b) = -C_0$ and there exists a stopping time $\tau \in \mathcal{T}_{[0, T]}$ such that

$$V_\tau(\phi^b) + Y_\tau \geq 0 \quad \text{and} \quad E^{P^*} [V_\tau(\phi^b) + Y_\tau] > 0. \tag{4.79}$$

In conditions (4.78)–(4.79), one could equivalently evaluate the expectations under P instead of P^* , since P and P^* have the same sets of probability

zero. The initial price C_0 is arbitrage free if there is no arbitrage opportunity for a seller or buyer of the contingent claim at this initial price.

To take advantage of a seller's arbitrage opportunity, an investor could sell one ACC at time zero for C_0 and invest the proceeds C_0 according to the trading strategy ϕ^s until the claim is cashed in by the buyer at some stopping time τ . At time τ , the seller would give the buyer Y_τ to pay off the claim and could put the remainder of his wealth $V_\tau(\phi^s) - Y_\tau$ in bond for the time period $(\tau, T]$. This would result in a final value of $(V_\tau(\phi^s) - Y_\tau)e^{r(T-\tau)}$, which is non-negative and is strictly positive with positive probability under P^* .

To take advantage of a buyer's arbitrage opportunity, an investor could buy one ACC at time zero for C_0 and invest $-C_0$ according to ϕ^b until the time τ when the buyer cashes in the claim. The buyer would then have $V_\tau(\phi^b) + Y_\tau$ at time τ and could put this in bond for the time period $(\tau, T]$ so that the buyer's final wealth would be $(V_\tau(\phi^b) + Y_\tau)e^{r(T-\tau)}$, which is non-negative and is strictly positive with positive probability under P^* .

Theorem 4.7.2. *The unique arbitrage free price at time zero for the American contingent claim is U_0 , which is defined by*

$$U_0 = \sup_{\tau \in \mathcal{T}_{[0, T]}} E^{P^*}[e^{-r\tau} Y_\tau]. \quad (4.80)$$

Proof. We first show that the arbitrage free price cannot be anything other than U_0 ; i.e., we establish uniqueness of an arbitrage free initial price. Let ϕ^* be the superhedging strategy defined in (4.74).

Suppose that $C_0 > U_0$. Then there is an arbitrage opportunity for the seller of the American contingent claim. To see this, let ϕ^s denote the self-financing trading strategy in stock and bond corresponding to investing U_0 according to the superhedging strategy ϕ^* and $C_0 - U_0$ in the bond for all time. Note that the initial value $V_0(\phi^s) = C_0$ and, by (4.76), the value $V_t(\phi^s)$ of ϕ^s at time t is at least as great as Y_t for each $t \in [0, T]$. Then for any stopping time $\tau \in \mathcal{T}_{[0, T]}$,

$$\begin{aligned} V_\tau(\phi^s) - Y_\tau &= V_\tau(\phi^*) + (C_0 - U_0)B_\tau - Y_\tau \\ &\geq Y_\tau + (C_0 - U_0)B_\tau - Y_\tau \\ &\geq (C_0 - U_0)B_\tau, \end{aligned}$$

where $(C_0 - U_0)B_\tau > 0$. Thus, there is an arbitrage opportunity for a seller of the American contingent claim.

On the other hand, suppose that $C_0 < U_0$. Then there is an arbitrage opportunity for the buyer of the American contingent claim. To see this, let ϕ^b denote the self-financing trading strategy corresponding to investing $-U_0$ according to the negative $-\phi^*$ of the superhedging strategy ϕ^* and investing

$U_0 - C_0$ in the bond for all time. Furthermore, consider the stopping time τ^* (viewed as the time at which an ACC, bought initially by the buyer, should be cashed in). Note that $V_t(-\phi^*) = -V_t(\phi^*)$ for $t \in [0, T]$, and $V_0(\phi^b) = -C_0$. Then, using the fact that $V_{\tau^*}(\phi^*) = U_{\tau^*} = Y_{\tau^*}$ (cf. (4.77)), we have

$$V_{\tau^*}(\phi^b) + Y_{\tau^*} = -V_{\tau^*}(\phi^*) + (U_0 - C_0)B_{\tau^*} + Y_{\tau^*} = (U_0 - C_0)B_{\tau^*}.$$

Since $(U_0 - C_0)B_{\tau^*} > 0$, there is an arbitrage opportunity for a buyer of the American contingent claim.

Finally, suppose that $C_0 = U_0$. We need to show that $C_0 = U_0$ is arbitrage free; i.e., that there exists an arbitrage free initial price. We begin by showing that there is no arbitrage opportunity for a seller of an American contingent claim with $C_0 = U_0$. For a proof by contradiction, suppose that there exists an admissible strategy ϕ^s such that $V_0(\phi^s) = U_0$ and for each $\tau \in \mathcal{T}_{[0, T]}$, (4.78) holds. Note that $\tau^* \in \mathcal{T}_{[0, T]}$. Then it follows from (4.78), with $\tau = \tau^*$, that $V_{\tau^*}(\phi^s) - Y_{\tau^*} \geq 0$ and strict inequality holds with positive probability under P^* . Multiplying through by $e^{-r\tau^*}$ and taking expectations, we see that

$$E^{P^*} [V_{\tau^*}(\phi^s) - Y_{\tau^*}] > 0. \quad (4.81)$$

On the other hand, by Doob's stopping theorem for continuous martingales (cf. Appendix C),

$$E^{P^*} [V_{\tau^*}(\phi^s)] = V_0(\phi^s) = U_0^*,$$

and by (ii) in Lemma 4.7.1, $E^{P^*} [Y_{\tau^*}] = U_0^*$. Combining these two properties, we obtain

$$E^{P^*} [V_{\tau^*}(\phi^s) - Y_{\tau^*}] = 0,$$

which contradicts (4.81). Therefore, no such ϕ^s exists, and consequently there is no arbitrage opportunity for a seller of the American contingent claim.

Next we show that there is no arbitrage opportunity for a buyer of the American contingent claim with $C_0 = U_0$. For a proof by contradiction, suppose that there exists an admissible strategy ϕ^b such that $V_0(\phi^b) = -U_0$ and a stopping time $\tau \in \mathcal{T}_{[0, T]}$ such that (4.79) holds. Then,

$$E^{P^*} [V_{\tau}(\phi^b) + Y_{\tau}] > 0.$$

However, by Doob's stopping theorem, $E^{P^*} [V_{\tau}(\phi^b)] = V_0(\phi^b) = -U_0^*$, and, by the definition of U_0^* , we have $E^{P^*} [Y_{\tau}] \leq U_0^*$. Therefore,

$$E^{P^*} [V_{\tau}(\phi^b) + Y_{\tau}] \leq 0,$$

which yields the desired contradiction. Hence there is no arbitrage opportunity for a buyer of the American contingent claim. $\square$

4.8. American Call Option

Consider an American call option with payoff process Y given by

$$Y_t = (S_t - K)^+, \quad t \in [0, T],$$

where K is a fixed constant in $(0, \infty)$. Note that by the Cauchy-Schwarz inequality and Doob's L^2 -martingale inequality,

$$\begin{aligned} E^{P^*} \left[\sup_{0 \leq t \leq T} Y_t \right] &\leq e^{rT} E^{P^*} \left[\sup_{0 \leq t \leq T} S_t^* \right] \\ &\leq e^{rT} \left(E^{P^*} \left[\sup_{0 \leq t \leq T} (S_t^*)^2 \right] \right)^{\frac{1}{2}} \\ &\leq 2e^{rT} \left(E^{P^*} [(S_T^*)^2] \right)^{\frac{1}{2}}, \\ &= 2S_0^2 e^{rT + \frac{1}{2}\sigma^2 T} \left(E^{P^*} \left[\exp \left(2\sigma \tilde{W}_T - \frac{1}{2}(4\sigma^2 T) \right) \right] \right)^{\frac{1}{2}} \\ &= 2S_0^2 e^{rT + \frac{1}{2}\sigma^2 T}. \end{aligned}$$

For the next to last inequality above we have used (4.33). Thus, Y is a non-negative, continuous, adapted process satisfying $E^{P^*}[\sup_{0 \leq t \leq T} Y_t] < \infty$, as required in the hypotheses of the previous section. Hence, by Theorem 4.7.2, the initial arbitrage free price is given by

$$U_0 = \sup_{\tau \in \mathcal{T}_{[0, T]}} E^{P^*} [Y_\tau^*], \quad (4.82)$$

$$= \sup_{\tau \in \mathcal{T}_{[0, T]}} E^{P^*} [e^{-r\tau} (S_\tau - K)^+]. \quad (4.83)$$

By part (ii) of Lemma 4.7.1, the supremum in the above expression is achieved by $\tau^* = \tau^*(0) = \inf\{t \in [0, T] : U_t^* = Y_t^*\}$. In general, neither U_0 nor τ^* is easy to compute. However, in this section, we will exploit the convexity and increasing property of the function $g(x) = (x - K)^+$, $x \geq 0$, to show that the initial arbitrage free price for an American call option is the same as that for the European call option with the same strike price and expiration time. One can also show that a buyer of the American call option who does not cash it in until the expiration time T will not create an arbitrage opportunity for the seller of the option. We leave this as an exercise for the reader.

We begin by deriving a stochastic integral equation for the discounted payoff process Y^* . For this, we use a couple of results from stochastic calculus that are beyond the summary provided in Appendix D. We provide references where these results are cited.

Under P^* , $\{S_t = e^{rt} S_t^*, t \geq 0\}$ is an Itô process satisfying the following equation P^* -a.s.:

$$S_t = S_0 + \int_0^t r S_s ds + \sigma \int_0^t S_s d\tilde{W}_s, \quad t \in [0, T]. \quad (4.84)$$

By the Itô-Tanaka formula (cf. Theorems VI.1.2 and VI.1.5 of Revuz-Yor [35]), P^* -a.s.,

$$Y_t = g(S_t) = g(S_0) + \int_0^t 1_{\{S_s > K\}} dS_s + \frac{1}{2} L_t, \quad t \in [0, T], \quad (4.85)$$

where $L = \{L_t, t \in [0, T]\}$ is the local time at K of the stock price process S . This local time is a continuous, non-decreasing, adapted process that measures the amount of time (in units of the quadratic variation of S) that S spends near K . Indeed, P^* -a.s., for each $t \in [0, T]$,

$$L_t = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^t 1_{[K, K+\epsilon)}(S_s) d[S]_s, \quad (4.86)$$

where $d[S]_s = \sigma^2 S_s^2 ds$; cf. Revuz and Yor [35], Corollary VI.1.9. (The Itô-Tanaka formula is a generalization of Itô's formula from twice continuously differentiable functions to convex functions. It can be derived by passing to a limit in Itô's formula applied to a sequence of smooth functions $\{g_n\}$ that approximate g in a suitable manner. The term in Itô's formula containing the first derivative of such a function converges to the second term after the second equality in (4.85). The term in Itô's formula containing the second derivative of such a function converges to the local time term in (4.85). See Chapter VI of [35] for details.)

Since $Y_t^* = e^{-rt} g(S_t)$, $t \in [0, T]$ is the product of a continuous, non-decreasing, adapted process and an Itô process, it obeys the usual product rule for differentiation. (For a justification of this rule of stochastic calculus,

see [11], Chapter 5.) Consequently we have P^* -a.s.,

$$\begin{aligned}
Y_t^* &= Y_0^* - \int_0^t r e^{-rs} g(S_s) ds \\
&\quad + \int_0^t e^{-rs} \left(1_{\{S_s > K\}} dS_s + \frac{1}{2} dL_s \right) \\
&= Y_0^* - \int_0^t r e^{-rs} g(S_s) ds \\
&\quad + \int_0^t e^{-rs} \left(1_{\{S_s > K\}} (r S_s ds + \sigma S_s d\tilde{W}_s) + \frac{1}{2} dL_s \right) \\
&= Y_0^* + \int_0^t e^{-rs} 1_{\{S_s > K\}} \sigma S_s d\tilde{W}_s \\
&\quad + \int_0^t e^{-rs} (1_{\{S_s > K\}} r S_s - r g(S_s)) ds + \frac{1}{2} \int_0^t e^{-rs} dL_s.
\end{aligned}$$

Here the integral with respect to L is defined pathwise as a continuous, adapted, non-decreasing process. Notice that by the definition of g , we have $1_{\{S_s > K\}} r S_s - r g(S_s) = 1_{\{S_s > K\}} r K$ for all $s \in [0, T]$. Thus, P^* -a.s., for $t \in [0, T]$,

$$\begin{aligned}
Y_t^* &= Y_0^* + \sigma \int_0^t e^{-rs} 1_{\{S_s > K\}} S_s d\tilde{W}_s \\
&\quad + r K \int_0^t e^{-rs} 1_{\{S_s > K\}} ds + \frac{1}{2} \int_0^t e^{-rs} dL_s.
\end{aligned}$$

The first integral above defines a continuous local martingale that starts from zero and the final two terms define continuous, adapted, non-decreasing processes. Moreover, Y_t^* is dominated by the P^* -integrable random variable $\sup_{t \in [0, T]} Y_t^*$ for each $t \in [0, T]$. It follows that $Y^* = \{Y_t^*, \mathcal{F}_t, t \in [0, T]\}$ is a continuous submartingale. Therefore, by Doob's stopping theorem (cf. Appendix C), for each $\tau \in \mathcal{T}_{[0, T]}$,

$$E^{P^*} [Y_\tau^*] \leq E^{P^*} [Y_T^*].$$

This together with (4.82) gives

$$U_0 = \sup_{\tau \in \mathcal{T}_{[0, T]}} E^{P^*} [Y_\tau^*] \leq E^{P^*} [Y_T^*].$$

But notice that $\tau \equiv T$ is an element of $\mathcal{T}_{[0, T]}$ and so by (4.82),

$$U_0 \geq E^{P^*} [Y_T^*].$$

It follows that the initial arbitrage free price for the American call option with strike price K and expiration time T is given by

$$U_0 = E^{P^*} [Y_T^*] = e^{-rT} E^{P^*} [(S_T - K)^+],$$

which is the same as the initial arbitrage free price of the European call option with the same strike price and expiration time.

4.9. American Put Option

Consider an American put option with strike price $K \in (0, \infty)$ and expiration time T . Then this has payoff process $Y = \{Y_t, t \in [0, T]\}$, where

$$Y_t = (K - S_t)^+, \quad t \in [0, T].$$

Since Y is a continuous, adapted process bounded by K , and $Y_t \geq 0$ for all $t \in [0, T]$, Y satisfies the hypotheses required of an American contingent claim in Section 4.7. Thus, by Theorem 4.7.2, the initial arbitrage free price for this American put option is given by

$$U_0 = \sup_{\tau \in \mathcal{T}_{[0, T]}} E^{P^*}[e^{-r\tau}(K - S_\tau)^+]. \quad (4.87)$$

The right member above is difficult to compute directly, as it involves taking a supremum over an uncountable family of stopping times. In fact, for each $t \in [0, T]$, the value of $U_t = e^{rt}U_t^*$, for U_t^* as defined in (4.70), can be characterized in terms of a solution of a free boundary problem for a parabolic partial differential equation. Here we shall motivate and state results associated with this characterization. However, for a detailed development including proofs, we refer the reader to the more advanced text on mathematical finance by Karatzas and Shreve [28], Section 2.7. Their treatment is largely based on the original paper of Jacka [24].

Recall from (4.33) and (4.10) that under P^* ,

$$S_t = S_0 \exp\left(\sigma \tilde{W}_t - \frac{1}{2}\sigma^2 t + rt\right), \quad t \in [0, T], \quad (4.88)$$

where $\tilde{W}$ is a standard Brownian motion. Now, for each $T \in [0, \infty)$ and $x \in [0, \infty)$, let $v(T, x)$ denote the value of the right member of (4.87) given that $S_0 = x$. Note that $v(0, x) = (K - x)^+$. (Although $T > 0$ has so far been fixed in this chapter and $S_0 > 0$, for the definition of $v(T, x)$ we allow T to vary in $[0, \infty)$ and allow $S_0 = 0$ as a possibility.)

Proposition 4.9.1. *The function $v : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ is continuous and satisfies $v(t, x) \geq (K - x)^+$ for all $t \in [0, \infty)$ and $x \in [0, \infty)$.*

Proof. See Proposition 7.1 in Section 2.7 of [28]. □

We now revert to $T \in (0, \infty)$ and $S_0 > 0$ being fixed. Then,

$$U_0 = v(T, S_0). \quad (4.89)$$

In fact, one can express the entire process $\{U_t, t \in [0, T]\}$, which turns out to be the arbitrage free price process for the American put (see the

Exercises), in terms of the function v as follows. For this, we note that the filtration $\{\mathcal{F}_t, t \in [0, T]\}$ is the same as that generated by the process $\{S_t, t \in [0, T]\}$ under P^* (this includes augmentation of the filtration by the P^* -null sets). By the stationary independent increments property of $\tilde{W}$ under P^* , $\{S_t, t \in [0, T]\}$ is a strong Markov process. In particular, using this property, it can be shown that P^* -a.s., for each $t \in [0, T]$,

$$\begin{aligned} U_t &= \text{ess sup}_{\tau \in \mathcal{T}_{[t, T]}} E^{P^*} \left[e^{-r(\tau-t)} (K - S_\tau)^+ \mid \mathcal{F}_t \right] \\ &= v(T - t, S_t). \end{aligned}$$

Recall the definition of τ^* from Section 4.7 and the fact that $\{U_{t \wedge \tau^*}^*, \mathcal{F}_t, t \in [0, T]\}$ is a martingale under P^* . Using the above we have

$$\tau^* = \inf\{s \geq 0 : v(T - s, S_s) = (K - S_s)^+\}. \quad (4.90)$$

This motivates introducing the continuation region

$$\mathcal{C} = \{(t, x) \in (0, \infty) \times (0, \infty) : v(t, x) > (K - x)^+\}$$

and the stopping region

$$\mathcal{D} = \{(t, x) \in [0, \infty) \times [0, \infty) : v(t, x) = (K - x)^+\}.$$

For each $t \in (0, \infty)$, define the section of $\mathcal{C}$ by

$$\mathcal{C}_t = \{x \in (0, \infty) : v(t, x) > (K - x)^+\}. \quad (4.91)$$

Proposition 4.9.2. *For each $t \in (0, \infty)$, there is a number $c(t) \in (0, K)$ such that $\mathcal{C}_t = (c(t), \infty)$. The function $c : (0, \infty) \rightarrow (0, K)$ is non-increasing, continuous, and $\lim_{t \downarrow 0} c(t) = K$. We denote this last quantity by $c(0)$.*

Proof. See Section 2.7 of [28]. $\square$

The function c defines the “free boundary”. It turns out that v satisfies a parabolic partial differential equation on $\mathcal{C}$. In fact, $v(s, x)$ is once continuously differentiable in s and twice continuously differentiable in x for $(s, x) \in \mathcal{C}$. For $(s, x) \in \mathcal{C}$, we let $v_s(s, x)$ denote the first partial derivative with respect to s and $v_x(s, x)$, $v_{xx}(s, x)$ denote the first, second partial derivatives with respect to x , respectively. Assuming that these partial derivatives exist and are continuous in $\mathcal{C}$, we now motivate the form of the partial differential equation satisfied by v .

For each $\epsilon > 0$, let

$$\tau_\epsilon^* = \inf\{s \geq 0 : v(T - s, S_s) \leq (K - S_s)^+ + \epsilon\}.$$

Since $v(0, x) = (K - x)^+$, and v and S are continuous, we have $\tau_\epsilon^* < T$. We can apply Itô's formula to the Itô process $\{(T - (t \wedge \tau_\epsilon^*), S_{t \wedge \tau_\epsilon^*}), t \in [0, T]\}$ and the function v defined on $\mathcal{C}$, noting that on $\{\tau_\epsilon^* > 0\}$, the Itô process has paths in $\mathcal{C}$ and that for Itô's formula we need only continuous first partial

derivatives with respect to the first variable in v as the first coordinate of the Itô process has paths of bounded variation (cf. [11], Theorem 5.10). Making use of (4.84), we obtain for each $\epsilon > 0$, P^* -a.s., for each $t \in [0, T]$,

$$\begin{aligned}
U_{t \wedge \tau_\epsilon^*} &= v(T - (t \wedge \tau_\epsilon^*), S_{t \wedge \tau_\epsilon^*}) \\
&= v(T, S_0) - \int_0^{t \wedge \tau_\epsilon^*} v_s(T - u, S_u) du \\
&\quad + \int_0^{t \wedge \tau_\epsilon^*} v_x(T - u, S_u) dS_u \\
&\quad + \frac{1}{2} \int_0^{t \wedge \tau_\epsilon^*} v_{xx}(T - u, S_u) \sigma^2 S_u^2 du \\
&= v(T, S_0) + \int_0^{t \wedge \tau_\epsilon^*} v_x(T - u, S_u) \sigma S_u d\tilde{W}_u \\
&\quad + \int_0^{t \wedge \tau_\epsilon^*} (\mathcal{L}v)(T - u, S_u) du,
\end{aligned}$$

where for $(s, x) \in \mathcal{C}$,

$$(\mathcal{L}v)(s, x) = -v_s(s, x) + rxv_x(s, x) + \frac{1}{2}\sigma^2 x^2 v_{xx}(s, x).$$

Hence, applying Itô's formula again, we have P^* -a.s., for each $t \in [0, T]$,

$$\begin{aligned}
U_{t \wedge \tau_\epsilon^*}^* &= e^{-r(t \wedge \tau_\epsilon^*)} U_{t \wedge \tau_\epsilon^*} \\
&= U_0^* + \int_0^{t \wedge \tau_\epsilon^*} e^{-ru} v_x(T - u, S_u) \sigma S_u d\tilde{W}_u \\
&\quad + \int_0^{t \wedge \tau_\epsilon^*} e^{-ru} (\tilde{\mathcal{L}}v)(T - u, S_u) du,
\end{aligned}$$

where for $(s, x) \in \mathcal{C}$,

$$(\tilde{\mathcal{L}}v)(s, x) = (\mathcal{L}v)(s, x) - rv(s, x). \quad (4.92)$$

Now, since $\tau_\epsilon^* \leq \tau^*$, by part (iii) of Lemma 4.7.1 and Doob's stopping theorem, we have that $\{U_{t \wedge \tau_\epsilon^*}^*, \mathcal{F}_t, t \in [0, T]\}$ is a right continuous martingale under P^* . Hence it follows by the uniqueness of its Doob-Meyer decomposition (cf. [27], Section 1.4) that P^* -a.s., for all $t \in [0, T]$:

$$\int_0^{t \wedge \tau_\epsilon^*} e^{-ru} (\tilde{\mathcal{L}}v)(T - u, S_u) du = 0.$$

One way in which this could be satisfied is if $\tilde{\mathcal{L}}v = 0$ on $\mathcal{C}$, since $(T - u, S_u) \in \mathcal{C}$ whenever $0 < u < \tau_\epsilon^*$. In fact, the following free boundary problem characterizes v , together with the boundary function c .

Proposition 4.9.3. *The pair $(f, d) = (v, c)$ is the unique pair of functions satisfying the following:*

- (i) $f : [0, \infty)^2 \rightarrow [0, \infty)$ is continuous and such that $f(t, x)$ is once continuously differentiable in t and twice continuously differentiable in x for all $(t, x) \in \mathcal{G} = \{(t, x) \in (0, \infty) \times (0, \infty) : x > d(t)\}$;
- (ii) $d : [0, \infty) \rightarrow (0, K]$ is non-increasing and left continuous, $d(0) = \lim_{x \downarrow 0} d(x) = K$;
- (iii)

$$\begin{aligned}
\tilde{\mathcal{L}}f &= 0 \quad \text{in } \mathcal{G}, \\
f(t, x) &\geq (K - x)^+, \quad (t, x) \in [0, \infty)^2, \\
f(t, x) &= K - x, \quad 0 \leq t < \infty, \quad 0 \leq x \leq d(t), \\
f(0, x) &= (K - x)^+ \quad \text{for } d(0) \leq x < \infty, \\
\lim_{x \rightarrow \infty} \max_{0 \leq s \leq t} |f(s, x)| &= 0, \quad t \in (0, \infty) \\
\lim_{x \downarrow d(t)} f_x(t, x) &= -1, \quad t \in (0, \infty).
\end{aligned}$$

Proof. See Theorem 7.12 of Section 2.7 in [28]. $\square$

While the above proposition provides a characterization of v and the free boundary c , this still does not provide a closed form expression for v . However, the above free boundary value problem can be expressed in variational form and that can be used as the basis for a numerical method for approximating v . See [28], Section 2.8, for more discussion of this and other methods for approximating v .

4.10. Exercises

For the following exercises, you should assume that the primary market model is given by the Black-Scholes model.

1. Verify that in the Example in Section 4.2, ϕ is not an admissible strategy.
2. Suppose that time is measured in years and that the riskless interest rate is 5% per year (compounded continuously). Assume that the stock moves according to a geometric Brownian motion with drift parameter $\mu = 0.15$, volatility parameter $\sigma = 0.20$ and a current price of \$10.
 - (a) Consider a European call option on the stock with an exercise price of \$12 and an expiration date of six months from now. What is the arbitrage free price for this option now? Describe a replicating strategy for this option.
 - (b) What is the current arbitrage free price for a European contingent claim that pays \$1 if the stock price is more than \$10 six months from now and pays off zero otherwise?

- (c) By numerically approximating solutions of stochastic differential equations, or otherwise, estimate the current arbitrage free price of a European contingent claim called an *Asian* option that has a value at its expiration six months from now of

$$\left(2 \int_0^{0.5} S_t dt - K \right)^+, \quad (4.93)$$

where S_t denotes the stock price t years from now and $K = \$12$.

3. Verify that C_0 given by the formula (4.62) is a strictly increasing function of σ for fixed K and T .
4. Consider a European call option with a strike price of \$70 and an expiration date of one month from now. The option is based on a stock with a current price of \$82. Suppose that the current price of the option is \$14. Assuming that the risk free interest rate is 6% per year, estimate the implied volatility of the stock. Please indicate the units in which your answer is measured. (You may use Newton's method or some other suitable numerical scheme. You will also need to use a reasonable approximation for the number of trading days between now and a month from now. Please use 21 days or 1/12 of a year for this.)
5. This exercise requires you to obtain current prices for options on IBM stock and historical data for the price of IBM stock. Such information can be readily obtained for example from the Yahoo Finance web site.

- (a) For this part of the exercise, you will need a list of current prices for call options on IBM stock. Consider the call options on IBM stock that expire next month. These all expire on the same day of that month but may have differing strike prices and differing initial prices. For each of these options, assuming a Black-Scholes model and that these are European options, estimate the implied volatility. (Assume that the 3-month T-bill rate is the risk free interest rate.) Make a graph of implied volatility versus strike price from these calculations.
- (b) This part of the exercise requires you to obtain historical data on the daily closing price of IBM stock. Assuming the Black-Scholes model, consider the approximation for the stock price S over the interval $[0, T]$ given by

$$S_{t+\Delta t} - S_t = \mu S_t \Delta t + \sigma S_t \Delta_t W, \quad (4.94)$$

where $\Delta t = T/n$ (some positive integer n), t is a multiple of Δt , $\Delta_t W = W_{t+\Delta t} - W_t$ is an increment of the Brownian motion W that is normally distributed with mean 0 and variance Δt . Here

the natural choice for Δt , given daily closing price data, is one day. (In our time line, we ignore non-trading days such as weekends and holidays.) For $i = 1, \dots, n \equiv \frac{T}{\Delta t}$, let

$$X_i = \frac{S_{i\Delta t} - S_{(i-1)\Delta t}}{S_{(i-1)\Delta t}}. \quad (4.95)$$

Then, the usual estimators for μ and σ^2 are given by

$$\bar{\mu} = \frac{1}{T} \sum_{i=1}^n X_i, \quad \bar{\sigma}^2 = \frac{1}{(n-1)\Delta t} \sum_{i=1}^n (X_i - \bar{\mu}\Delta t)^2. \quad (4.96)$$

Apply the above to the closing stock price data for IBM stock over a one-month period ending today. Compare the estimate $\bar{\sigma}$ of the volatility obtained in this way with the implied volatility estimates obtained from the option prices.

6. Let $X \in \mathcal{F}_T$ represent the final value of a European contingent claim and assume that $E^{P^*}[|X|] < \infty$. Verify that strategies of the form (4.46) satisfy the conditions (i)–(v) of Section 4.5 when the price process C for the European contingent claim is a continuous modification of $\{e^{rt}E^{P^*}[X^*|\mathcal{F}_t], t \in [0, T]\}$.

7. In a manner similar to that in Section 4.6, derive formulas for the arbitrage free price process and a replicating strategy associated with a European put option having a strike price of K and an expiration time of T .

8. Consider an American contingent claim and the associated process $\{U_t^*, t \in [0, T]\}$ as defined in Section 4.7. Prove that $\{U_t = e^{rt}U_t^*, t \in [0, T]\}$ is the unique arbitrage free price process for the American contingent claim. (For this, you will need to develop a notion of arbitrage for trading in stock, bond and the American contingent claim at all times $t \in [0, T]$. You should make sure to indicate this definition; cf. Section 4.5.)

9. Consider an American call option that can be traded only at time zero. Show that a buyer who purchases the American call option for its arbitrage free initial price and who does not exercise the option before time T will prevent the seller of the option from making a risk free profit (i.e., there will be no seller's arbitrage).