

Finite Market Model

The binomial model considered in the previous chapter is an example of a finite market model. In that example, we saw that the existence of both a risk neutral probability and a replicating strategy played a key role in justifying the unique arbitrage free price for any European contingent claim. In this chapter, we extend that idea to the pricing of European contingent claims in a general finite market model. We characterize those finite market models in which there is a risk neutral probability and in which all European contingent claims can be replicated. Indeed, we will prove the first fundamental theorem of asset pricing, which shows the equivalence of the absence of arbitrage in a finite market model to the existence of a risk neutral probability. We will also prove the second fundamental theorem of asset pricing, which shows that all European contingent claims in a finite market model without arbitrage can be replicated if and only if there is a unique risk neutral probability. Then, assuming there is such a unique risk neutral probability and allowing trading of the European contingent claim at all times, we show that there is a unique arbitrage free price process for each European contingent claim. These results have their origins in the 1979 seminal paper of Harrison and Kreps [20]. In the binomial model, there is a unique risk neutral probability and hence there is a unique arbitrage free price process for every European contingent claim (cf. Exercise 4 of Chapter 2). We round out this chapter with a discussion of (single period) markets in which there is more than one risk neutral probability. Although such markets have no arbitrage opportunities, there are some European contingent claims that are not replicable; these are called *incomplete* markets.

3.1. Definition of the Finite Market Model

Throughout this chapter we will be working within the framework of the following discrete time, finite state market model. For short we will call this a finite market model.

Let $(\Omega, \mathcal{F}, P)$ be a probability space where Ω is a finite set of possible outcomes, $\mathcal{F}$ is the σ -algebra consisting of all subsets of Ω and P is a probability measure on $(\Omega, \mathcal{F})$ such that $P(\{\omega\}) > 0$ for all $\omega \in \Omega$. Expectations under P will be written simply as $E[\cdot]$. Whenever another probability is to be used, this will be explicitly indicated in the notation.

We assume that there are finitely many times $t = 0, 1, 2, \dots, T$ (where $T < \infty$ is a positive integer and successive times are successive integers). At each of these times the values of $d + 1$ assets can be observed. Here d is a strictly positive integer. One asset is a riskless security called a bond, and the other d assets are risky securities called stocks.

A σ -algebra $\mathcal{F}_t \subset \mathcal{F}$ describes the information available to an investor at time t . It is assumed that $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_T$ and $\mathcal{F}_T = \mathcal{F}$. The collection $\{\mathcal{F}_t, t = 0, 1, \dots, T\}$ is called a filtration.

The *bond* (asset labelled 0) is assumed to have price process $S^0 = \{S_t^0, t = 0, 1, \dots, T\}$, where S_t^0 denotes the price of the bond at time t . We assume that for each t , $S_t^0 > 0$ and S_t^0 is deterministic (i.e., $S_t^0 \in \mathcal{F}_0$). For example, if the bond has an interest rate of $r \geq 0$ per unit of time, then $S_t^0 = (1 + r)^t$ for all t . The bond is considered to be a “numeraire”; i.e., it tells us what a dollar at time 0 is worth (due, amongst other things, to the effects of inflation) at time t .

The d *stocks* are assumed to have price processes $S^1, \dots, S^d$, where S_t^i is the price of the i^{th} stock at time t . It is assumed that S_t^i is an $\mathcal{F}_t$ -measurable random variable for $i = 1, \dots, d$ and $t = 0, 1, \dots, T$. Note that since Ω is finite, $S_t = (S_t^0, S_t^1, \dots, S_t^d)$, $t = 0, 1, \dots, T$, can take on at most finitely many values. It follows that in the development below, all of the expectations we write will be automatically finite.

A *trading strategy* (in the finite market model) is a collection of $(d + 1)$ -dimensional vectors indexed by $t = 1, \dots, T$:

$$\phi = \{\phi_t, t = 1, \dots, T\}, \quad (3.1)$$

where for each $t \in \{1, \dots, T\}$, $\phi_t = (\phi_t^0, \phi_t^1, \dots, \phi_t^d)$ is such that ϕ_t^i is a real-valued $\mathcal{F}_{t-1}$ -measurable random variable for $i = 0, 1, \dots, d$. We regard ϕ_t^i as representing the number of “shares” of asset i to be held over the time interval $(t - 1, t]$. In particular, ϕ_t^0 denotes the number of units of the bond to be held over this interval, and ϕ_t^i denotes the number of shares of stock i to be held over the interval for $i = 1, \dots, d$. A positive value for ϕ_t^i indicates

that one buys that number of shares of asset i , at a price of S_{t-1}^i per share, and holds them over the interval $(t-1, t]$. A negative value for ϕ_t^i indicates that asset i will be “sold short”. For example, if $\phi_t^i = -1$, one is effectively borrowing the value S_{t-1}^i of asset i at time $t-1$ with the understanding that the cost to repay this loan at time t is the value of one share of asset i at time t ; i.e., S_t^i . We will restrict attention to *self-financing* trading strategies, namely, those trading strategies ϕ such that the investor’s initial wealth is given by

$$\phi_1 \cdot S_0, \quad (3.2)$$

and

$$\phi_t \cdot S_t = \phi_{t+1} \cdot S_t, \quad t = 1, \dots, T-1, \quad (3.3)$$

where $\cdot$ denotes the dot product in $\mathbb{R}^{d+1}$. In this chapter, the term “trading strategy” will always mean self-financing trading strategy.

The *initial value* of a trading strategy ϕ is $V_0(\phi) = \phi_1 \cdot S_0$ and its *value* at time $t \in \{1, \dots, T\}$ is

$$V_t(\phi) \equiv \phi_t \cdot S_t = \sum_{i=0}^d \phi_t^i S_t^i. \quad (3.4)$$

Using the self-financing property we also have that

$$V_t(\phi) = \phi_{t+1} \cdot S_t, \quad t = 0, 1, \dots, T-1. \quad (3.5)$$

The *gains process* associated with a trading strategy ϕ is defined by

$$G_t(\phi) = V_t(\phi) - V_0(\phi), \quad t = 0, 1, \dots, T. \quad (3.6)$$

Using the equivalent forms for $V_s(\phi)$ that come from the self-financing property of ϕ , we can rewrite this process as follows for $t = 1, \dots, T$:

$$\begin{aligned} G_t(\phi) &= \sum_{s=1}^t (V_s(\phi) - V_{s-1}(\phi)) \\ &= \sum_{s=1}^t (\phi_s \cdot S_s - \phi_s \cdot S_{s-1}) \\ &= \sum_{s=1}^t \phi_s \cdot (S_s - S_{s-1}) \\ &= \sum_{s=1}^t \phi_s \cdot \Delta S_s, \end{aligned} \quad (3.7)$$

where $\Delta S_s \equiv S_s - S_{s-1}$. In fact, the last expression is a discrete time stochastic integral (recall that ϕ_s is $\mathcal{F}_{s-1}$ -measurable).

An *arbitrage opportunity* (in the finite market model) is a trading strategy ϕ such that

$$V_0(\phi) = 0, \quad V_T(\phi) \geq 0, \quad E[V_T(\phi)] > 0.$$

The finite market model is said to be *viable* if it has no arbitrage opportunities.

It will simplify computations to use *discounted asset price processes*, obtained by normalizing so that the value of a dollar at any time t is the same as it is at time 0. This is often called a change of numeraire. For $i = 0, 1, \dots, d$, we define

$$S_t^{*,i} = \frac{S_t^i}{S_t^0}, \quad \text{for } t = 0, 1, \dots, T.$$

Note that $S_t^{*,0} \equiv 1$ for all t . Then $S_t^* = (S_t^{*,0}, S_t^{*,1}, \dots, S_t^{*,d})$ is the value of the vector of discounted asset prices at time t . We will refer to $S^* = \{S_t^*, t = 0, 1, \dots, T\}$ as the (vector) discounted asset price process. The associated *discounted value process* for a trading strategy ϕ is defined by

$$V_t^*(\phi) \equiv \frac{V_t(\phi)}{S_t^0}, \quad t = 0, 1, \dots, T, \quad (3.8)$$

and using (3.4) and (3.5) we see that

$$V_t^*(\phi) = \phi_t \cdot S_t^*, \quad t = 1, \dots, T, \quad (3.9)$$

and

$$V_t^*(\phi) = \phi_{t+1} \cdot S_t^*, \quad t = 0, 1, \dots, T-1. \quad (3.10)$$

The discounted gains process for ϕ is defined by

$$G_t^*(\phi) = V_t^*(\phi) - V_0^*(\phi), \quad t = 0, 1, \dots, T, \quad (3.11)$$

and by very similar manipulations to those used in deriving (3.7), this can be reexpressed as $G_0^* = 0$ and

$$G_t^*(\phi) = \sum_{s=1}^t \phi_s \cdot \Delta S_s^*, \quad t = 1, \dots, T, \quad (3.12)$$

where $\Delta S_s^* = S_s^* - S_{s-1}^*$. An advantage of this last expression is that it involves only the risky assets, since $\Delta S_s^{*,0} = 0$ for $s = 1, \dots, T$.

3.2. First Fundamental Theorem of Asset Pricing

The following definitions will be needed to state the first fundamental theorem of asset pricing, which characterizes viable finite market models.

Definition 3.2.1. Two probability measures Q and Q' on $(\Omega, \mathcal{F})$ are equivalent (or mutually absolutely continuous) provided for each $A \in \mathcal{F}$,

$$Q(A) = 0 \quad \text{if and only if} \quad Q'(A) = 0. \quad (3.13)$$

Remark. In the finite market model, P gives positive probability to every $\omega \in \Omega$, and so for a probability measure P^* on $(\Omega, \mathcal{F})$, P is equivalent to P^* if and only if $P^*(\{\omega\}) > 0$ for all $\omega \in \Omega$.

Definition 3.2.2. An equivalent martingale measure (abbreviated as EMM) is a probability measure P^* defined on $(\Omega, \mathcal{F})$ such that P^* is equivalent to P and S^* is a martingale under P^* (relative to the filtration $\{\mathcal{F}_t, t = 0, 1, \dots, T\}$); i.e., for each $t \in \{1, \dots, T\}$,

$$E^{P^*}[S_t^* | \mathcal{F}_{t-1}] = S_{t-1}^*, \quad (3.14)$$

where $E^{P^*}[\cdot]$ denotes expectation under P^* , and the above equality is to be interpreted componentwise.

We note for future use that (3.14) is equivalent to

$$E^{P^*}[\Delta S_t^* | \mathcal{F}_{t-1}] = 0. \quad (3.15)$$

Remark. An equivalent martingale measure is sometimes also called a *risk neutral probability*. We will use the terms interchangeably.

Lemma 3.2.3. Suppose that P^* is an equivalent martingale measure and ϕ is a trading strategy in the finite market model. Then $\{V_t^*(\phi), \mathcal{F}_t, t = 0, 1, \dots, T\}$ is a martingale under P^* .

Proof. For $t = 1, \dots, T$, $V_t^*(\phi) = \phi_t \cdot S_t^* \in \mathcal{F}_t$ and for $t = 0$, $V_0^*(\phi) = \phi_1 \cdot S_0^* \in \mathcal{F}_0$, since $\phi_t \in \mathcal{F}_{t-1}$ for $t = 1, \dots, T-1$ and $S_t^* \in \mathcal{F}_t$ for all t . For each t , $V_t^*(\phi)$ is integrable because Ω is a finite set. Finally, for each $t \in \{1, 2, \dots, T\}$,

$$\begin{aligned} E^{P^*}[V_t^*(\phi) | \mathcal{F}_{t-1}] &= \phi_t \cdot E^{P^*}[S_t^* | \mathcal{F}_{t-1}] \\ &= \phi_t \cdot S_{t-1}^* = V_{t-1}^*(\phi), \end{aligned}$$

where we have used the fact that $\phi_t \in \mathcal{F}_{t-1}$, the martingale property (3.14) of S^* and the self-financing property (3.5) of ϕ . $\square$

Theorem 3.2.4. (First Fundamental Theorem of Asset Pricing) The finite market model is viable if and only if there exists an equivalent martingale measure P^* .

Proof. We first prove the “if” part of the theorem. Suppose there exists an equivalent martingale measure P^* . For a proof by contradiction, suppose that ϕ is an arbitrage opportunity; that is, ϕ is a trading strategy with initial

value $V_0(\phi) = 0$, final value $V_T(\phi) \geq 0$, and $E[V_T(\phi)] > 0$. It follows that the discounted values satisfy $V_0^*(\phi) = 0$, $V_T^*(\phi) \geq 0$, and since P^* is equivalent to P , $E^{P^*}[V_T^*(\phi)] > 0$. By Lemma 3.2.3, $\{V_t^*(\phi), \mathcal{F}_t, t = 0, 1, \dots, T\}$ is a martingale under P^* and so

$$E^{P^*}[V_T^*(\phi)] = E^{P^*}[V_0^*(\phi)].$$

However, the left member above is strictly positive whereas the right member is zero, which yields the desired contradiction. Thus, there cannot be an arbitrage opportunity in the finite market model, and hence the model is viable.

We now turn to proving the “only if” part of the theorem. For this, suppose that the finite market model is viable. Since Ω is a finite set, for any random variable Y defined on $(\Omega, \mathcal{F})$, by enumerating Ω as $\{\omega_1, \dots, \omega_n\}$, we may view Y as $(Y(\omega_1), \dots, Y(\omega_n)) \in \mathbb{R}^n$. Since $\mathcal{F}$ consists of all subsets of Ω , any point in $\mathbb{R}^n$ can be thought of as representing a real-valued random variable on $(\Omega, \mathcal{F})$. Thus, there is a one-to-one correspondence between points in $\mathbb{R}^n$ and (real-valued) random variables defined on Ω . Adopting this point of view for the terminal discounted gain random variables $G_T^*(\phi)$, we define

$$L = \{G_T^*(\phi) : \phi \text{ is a trading strategy such that } V_0(\phi) = 0\}.$$

Note that L is a linear space, since $G_T^*(\phi)$ is linear in ϕ and any linear combination of trading strategies with initial values of zero is again a trading strategy with the same initial value. Also, L is non-empty because the origin is contained in L . Let

$$D = \{Y \in \mathbb{R}^n : Y_i \geq 0 \text{ for } i = 1, \dots, n \text{ and } Y_j > 0 \text{ for some } j\}.$$

Thus, D is the positive orthant in $\mathbb{R}^n$ with the origin removed.

Since the market is assumed to be viable, $L \cap D = \emptyset$. To see this, observe that if the latter were not true, there would be a trading strategy ϕ with $V_0(\phi) = 0$, $V_T^*(\phi) = G_T^*(\phi) \geq 0$ and $V_T^*(\phi)(\omega_i) > 0$ for at least one i , which would represent an arbitrage opportunity. Let

$$F = \left\{ Y \in D : \sum_{i=1}^n Y_i = 1 \right\}. \quad (3.16)$$

Then F is a convex, compact, non-empty subset of $\mathbb{R}^n$ and $L \cap F = \emptyset$ because $L \cap D = \emptyset$.

By applying the Separating Hyperplane Theorem 3.6.1 (see Section 3.6 for a statement and proof of this theorem), we see that there is a vector $Z \in \mathbb{R}^n \setminus \{0\}$ such that the hyperplane $H = \{Y \in \mathbb{R}^n : Y \cdot Z = 0\}$ contains L and $Z \cdot Y > 0$ for all $Y \in F$. By setting $Y_i = 1$ if $i = j$ and $Y_i = 0$ if $i \neq j$,

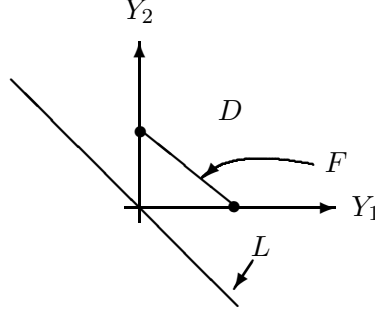


Figure 1. An example of the situation for $n = 2$

we see that $Z_j > 0$ for each $j \in \{1, \dots, n\}$. Define

$$P^*(\{\omega_i\}) = \frac{Z_i}{\sum_{j=1}^n Z_j}, \quad i = 1, \dots, n. \quad (3.17)$$

Then P^* is a probability measure on $(\Omega, \mathcal{F})$ and it is equivalent to P . Moreover, for any trading strategy ϕ such that $V_0(\phi) = 0$, we have

$$\begin{aligned} E^{P^*} [G_T^*(\phi)] &= \sum_{i=1}^n G_T^*(\phi)(\omega_i) \frac{Z_i}{\sum_{j=1}^n Z_j} \\ &= \frac{G_T^*(\phi) \cdot Z}{\sum_{j=1}^n Z_j} \\ &= 0, \end{aligned} \quad (3.18)$$

where the last line follows from the fact that Z is perpendicular to H , which contains L .

Note that $G_T^*(\phi)$ involves only $(\phi^1, \dots, \phi^d)$. From Lemma 3.2.5, proved below, given $\hat{\phi}^1, \dots, \hat{\phi}^d$, where for $i = 1, \dots, d$, $\hat{\phi}^i = \{\hat{\phi}_t^i, t = 1, \dots, T\}$ and $\hat{\phi}_t^i$ is a real-valued, $\mathcal{F}_{t-1}$ -measurable random variable for each t , there is a unique time-ordered set of T real-valued random variables $\phi^0 = \{\phi_t^0, t = 1, \dots, T\}$ such that $\phi \equiv \{(\phi_t^0, \hat{\phi}_t^1, \dots, \hat{\phi}_t^d), t = 1, \dots, T\}$ is a trading strategy with an initial value of zero. Upon substituting this in (3.18) and writing out the expression (cf. (3.12)) for $G_T^*(\phi)$, we see that

$$\begin{aligned} 0 = E^{P^*} [G_T^*(\phi)] &= E^{P^*} \left[\sum_{t=1}^T \phi_t \cdot \Delta S_t^* \right] \\ &= E^{P^*} \left[\sum_{t=1}^T \sum_{i=1}^d \hat{\phi}_t^i \Delta S_t^{*,i} \right]. \end{aligned} \quad (3.19)$$

For each fixed $i \in \{1, \dots, d\}$, if we set $\hat{\phi}_t^j = 0$ for all t and $j \neq i$, we obtain

$$0 = E^{P^*} \left[\sum_{t=1}^T \phi_t^i \Delta S_t^{*,i} \right], \quad (3.20)$$

for each $\hat{\phi}^i = \{\hat{\phi}_t^i, t = 1, \dots, T\}$ such that $\hat{\phi}_t^i$ is a real-valued $\mathcal{F}_{t-1}$ -measurable random variable for each t . It then follows from Lemma 3.2.6, proved below, that for $i = 1, \dots, d$, $S^{*,i}$ is a martingale under P^* . Hence, P^* is an equivalent martingale measure. $\square$

The next two lemmas were used in the above proof of the first fundamental theorem of asset pricing. The first lemma shows that given (non-anticipating) holdings in the risky assets and an initial wealth, there is a unique sequence of holdings in the riskless asset that makes the associated trading strategy self-financing.

Lemma 3.2.5. *For $i = 1, \dots, d$, let $\hat{\phi}^i = \{\hat{\phi}_t^i, t = 1, \dots, T\}$ where $\hat{\phi}_t^i$ is a real-valued, $\mathcal{F}_{t-1}$ -measurable random variable for $t = 1, \dots, T$. For each real-valued $\mathcal{F}_0$ -measurable random variable V_0 , there exists a unique time-ordered set of T real-valued random variables $\phi^0 = \{\phi_t^0, t = 1, \dots, T\}$ such that $\phi \equiv \{(\phi_t^0, \hat{\phi}_t^1, \dots, \hat{\phi}_t^d), t = 1, \dots, T\}$ is a trading strategy with an initial value of V_0 .*

Proof. Fix $V_0 \in \mathcal{F}_0$. For ϕ to be self-financing at time zero, we must have (cf. (3.2)):

$$\phi_1 \cdot S_0 = V_0 \quad (3.21)$$

and since $\hat{\phi}_1^1, \dots, \hat{\phi}_1^d$ are given, this will be satisfied if and only if

$$\phi_1^0 = (S_0^0)^{-1} \left(V_0 - \sum_{i=1}^d \hat{\phi}_1^i S_0^i \right). \quad (3.22)$$

Note that this $\phi_1^0 \in \mathcal{F}_0$. Thus, ϕ_1^0 is uniquely determined. For an induction, suppose that for some $1 \leq s \leq T-1$, $\phi_t^0, t = 1, \dots, s$, have been determined uniquely such that $\phi_t^0 \in \mathcal{F}_{t-1}$ for each $t = 1, \dots, s$, (3.21) holds, and

$$\phi_t \cdot S_t = \phi_{t+1} \cdot S_t, \quad t = 1, \dots, s-1. \quad (3.23)$$

Then, the self-financing property (3.23) holds for $t = s$ if and only if we have

$$\phi_{s+1}^0 = (S_s^0)^{-1} \left(\phi_s \cdot S_s - \sum_{i=1}^d \hat{\phi}_{s+1}^i S_s^i \right). \quad (3.24)$$

Note that this expression for ϕ_{s+1}^0 is $\mathcal{F}_s$ -measurable. This establishes the induction step, and it follows that there is a unique ϕ^0 that makes ϕ a trading strategy with initial value V_0 . $\square$

Lemma 3.2.6. *Let $M = \{M_t, t = 0, 1, \dots, T\}$ be a real-valued process such that $M_t \in \mathcal{F}_t$ for each t . Then, M is a martingale (relative to the filtration $\{\mathcal{F}_t\}$) if and only if*

$$E \left[\sum_{t=1}^T \eta_t \Delta M_t \right] = 0 \quad (3.25)$$

for all $\eta = \{\eta_t, t = 1, \dots, T\}$ such that η_t is a real-valued $\mathcal{F}_{t-1}$ -measurable random variable for $t = 1, \dots, T$. Here, $\Delta M_t = M_t - M_{t-1}$ for $t = 1, \dots, T$.

Remark. The sum $\sum_{t=1}^T \eta_t \Delta M_t$ is actually a discrete stochastic integral. If one extends η to a continuous time process by making it constant on $(t-1, t]$ with a value equal to that of η_t there, and one extends M to be constant on $[t-1, t)$ with a value equal to that of M_{t-1} there, then the sum is the same as the stochastic integral $\int_{[0, T]} \eta_t dM_t$.

Proof. Suppose M is a martingale. Let $\eta = \{\eta_t, t = 1, \dots, T\}$ where η_t is a real-valued $\mathcal{F}_{t-1}$ -measurable random variable for each t . Then, since $\eta_t \in \mathcal{F}_{t-1}$, we have

$$E \left[\sum_{t=1}^T \eta_t \Delta M_t \right] = \sum_{t=1}^T E [\eta_t E [\Delta M_t | \mathcal{F}_{t-1}]]. \quad (3.26)$$

Now, because M is a martingale we have $E [\Delta M_t | \mathcal{F}_{t-1}] = 0$, for $t = 1, \dots, T$, and it follows that (3.25) holds.

Conversely, suppose that (3.25) holds for all $\eta = \{\eta_t, t = 1, \dots, T\}$ where η_t is a real-valued $\mathcal{F}_{t-1}$ -measurable random variable for each t . For fixed $s \in \{1, \dots, T\}$ and $A \in \mathcal{F}_{s-1}$, let

$$\eta_t = \begin{cases} 0 & \text{for } t \neq s, \\ 1_A & \text{for } t = s. \end{cases}$$

Then $\eta_t \in \mathcal{F}_{t-1}$ for each t . Upon substituting this into (3.25), we obtain

$$E [1_A \Delta M_s] = 0. \quad (3.27)$$

Since $A \in \mathcal{F}_{s-1}$ was arbitrary, it follows that

$$E [M_s | \mathcal{F}_{s-1}] = M_{s-1}, \quad (3.28)$$

and then since s was arbitrary, it follows that M is a martingale. $\square$

3.3. Second Fundamental Theorem of Asset Pricing

A *European contingent claim* is represented by an $\mathcal{F}_T$ -measurable random variable X . This is interpreted to mean that the value (or payoff) of the contingent claim at the expiration time T is given by X . For example, a

European call option with strike price $K \in (0, \infty)$ and expiration date T that is based on the risky asset with price process S^1 is represented by $X = (S_T^1 - K)^+$. On the other hand, a look-back option usually depends on the recent history of a risky asset (see the Exercises for an example of such).

We will frequently refer to the random variable X , which represents a European contingent claim, as the European contingent claim (or ECC for short). For a European contingent claim X , we let $X^* = X/S_T^0$, the discounted value of X .

A *replicating (or hedging) strategy* for a European contingent claim X is a trading strategy ϕ such that $V_T(\phi) = X$. If there exists such a replicating strategy, the European contingent claim is said to be *attainable*.

The finite market model is said to be *complete* if all European contingent claims are attainable.

Theorem 3.3.1. *Suppose that the finite market model is viable and X is a replicable European contingent claim. Then the value process $\{V_t(\phi), t = 0, 1, \dots, T\}$ is the same for all replicating strategies ϕ for X . Indeed, for any replicating strategy ϕ and any equivalent martingale measure P^* , we have*

$$V_t^*(\phi) = E^{P^*}[X^* | \mathcal{F}_t], \quad t = 0, 1, \dots, T. \quad (3.29)$$

The right member of (3.29) defines the same stochastic process for all such P^ .*

Remark. For any equivalent martingale measure P^* , P and P^* have the same sets of probability zero. Now, the only set of zero probability under P is the empty set, and so it follows that statements in this chapter that normally would hold only P -a.s., or P^* -a.s., in fact hold surely. In particular, two stochastic processes are indistinguishable (see Appendix B) under P if and only if they are indistinguishable under P^* if and only if they are equal surely. This accounts for the fact that in the statement of the above theorem, we did not include any a.s. qualifiers.

Proof. By the first fundamental theorem of asset pricing, there is at least one equivalent martingale measure. Let P^* be such a measure. Let ϕ be a replicating strategy for X . Then, by the martingale property of $V^*(\phi)$ under P^* (see Lemma 3.2.3) and since $V_T^*(\phi) = X^*$, we have for $t = 0, 1, \dots, T$,

$$\begin{aligned} V_t^*(\phi) &= E^{P^*}[V_T^*(\phi) | \mathcal{F}_t] \\ &= E^{P^*}[X^* | \mathcal{F}_t]. \end{aligned} \quad (3.30)$$

Since the last expression above does not depend upon ϕ , it follows that $\{V_t^*(\phi), t = 0, 1, \dots, T\}$ and hence $\{V_t(\phi), t = 0, 1, \dots, T\}$ does not vary with the particular choice of replicating strategy ϕ . Furthermore, since the

left member of (3.30) does not depend upon the particular choice of an EMM P^* , it follows that a stochastic process defined by the right member of (3.30) does not depend on the particular choice of P^* . $\square$

Theorem 3.3.2. (*Second Fundamental Theorem of Asset Pricing*) *A viable finite market model is complete if and only if it admits a unique equivalent martingale measure.*

Proof. Suppose the market is viable and complete. By the first fundamental theorem of asset pricing, there is an equivalent martingale measure Q . Suppose there is another equivalent martingale measure $\tilde{Q}$. Fix $A \in \mathcal{F}_T$ and let $X = 1_A$. Suppose that ϕ is a replicating strategy for X . Then, by Theorem 3.3.1,

$$E^Q[X^*] = E^{\tilde{Q}}[X^*]. \quad (3.31)$$

Multiplying both sides by the deterministic quantity S_T^0 yields

$$E^Q[X] = E^{\tilde{Q}}[X] \quad (3.32)$$

and so

$$Q(A) = \tilde{Q}(A). \quad (3.33)$$

Hence, $Q = \tilde{Q}$, since $A \in \mathcal{F}_T = \mathcal{F}$ was arbitrary.

Conversely, suppose the market is viable but *not* complete. Then we will show that there is more than one equivalent martingale measure. Since the market is not complete, there exists a European contingent claim X that is not attainable. Let $\mathcal{P}$ denote the set of $\hat{\phi} = (\hat{\phi}^1, \dots, \hat{\phi}^d)$ satisfying $\hat{\phi}^i = \{\hat{\phi}_t^i, t = 1, \dots, T\}$ where $\hat{\phi}_t^i$ is a real-valued $\mathcal{F}_{t-1}$ -measurable random variable for $i = 1, \dots, d$, $t = 1, \dots, T$. We claim that there is no pair $(c, \hat{\phi})$ such that $\hat{\phi} \in \mathcal{P}$, $c \in \mathbb{R}$ and

$$c + \sum_{t=1}^T \hat{\phi}_t \cdot \Delta \hat{S}_t^* = X^*, \quad (3.34)$$

where $\hat{S}^* = (S^{*,1}, \dots, S^{*,d})$. To see this, observe that if there were such a pair $(c, \hat{\phi})$, then by Lemma 3.2.5, $\hat{\phi}$ could be extended to a trading strategy $\phi = (\phi^0, \hat{\phi}^1, \dots, \hat{\phi}^d)$ with initial value cS_0^0 , and then, by (3.11)–(3.12), ϕ would be a replicating strategy for X , which yields a contradiction.

Now, adopting the same device as in the proof of Theorem 3.2.4 of viewing random variables as vectors in $\mathbb{R}^n$, let

$$L = \left\{ c + \sum_{t=1}^T \hat{\phi}_t \cdot \Delta \hat{S}_t^* : \hat{\phi} \in \mathcal{P}, c \in \mathbb{R} \right\}. \quad (3.35)$$

Then, L is a linear subspace of $\mathbb{R}^n$ (since c is allowed to take any value in $\mathbb{R}$) and $X^* \notin L$. It follows that L is a strict subspace of $\mathbb{R}^n$ and there is a

vector $Z \in \mathbb{R}^n \setminus \{0\}$ such that $Z \in L^\perp$, the orthogonal complement of L in $\mathbb{R}^n$. Then,

$$\sum_{\omega \in \Omega} Z(\omega)Y(\omega) = 0 \quad \text{for all } Y \in L. \quad (3.36)$$

Since the finite market model is viable, there is at least one equivalent martingale measure P^* . Then $P^*(\{\omega\}) > 0$ for all $\omega \in \Omega$, and on setting

$$\tilde{Z}(\omega) = \frac{Z(\omega)}{P^*(\omega)} \quad \text{for all } \omega \in \Omega, \quad (3.37)$$

we may rewrite (3.36) as

$$E^{P^*}[\tilde{Z}Y] = 0 \quad \text{for all } Y \in L. \quad (3.38)$$

Now, define

$$P^{**}(\{\omega\}) = \left(1 + \frac{\tilde{Z}(\omega)}{2\|\tilde{Z}\|_\infty}\right) P^*(\{\omega\}) \quad \text{for each } \omega \in \Omega, \quad (3.39)$$

where $\|\tilde{Z}\|_\infty = \max_{\omega \in \Omega} |\tilde{Z}(\omega)|$. Since $\tilde{Z} \neq 0$, $P^{**} \neq P^*$. Moreover, $P^{**}(\{\omega\}) > 0$ for each $\omega \in \Omega$. To check that P^{**} is a probability measure, note that

$$\begin{aligned} P^{**}(\Omega) &= P^*(\Omega) + \sum_{\omega \in \Omega} \frac{\tilde{Z}(\omega)}{2\|\tilde{Z}\|_\infty} P^*(\{\omega\}) \\ &= 1 + \frac{1}{2\|\tilde{Z}\|_\infty} E^{P^*}[\tilde{Z}] \\ &= 1, \end{aligned}$$

where we used (3.38) with $Y = 1 \in L$ to obtain the last line. Thus, P^{**} is a probability measure that is equivalent to P^* . We finally need to check that S^* is a martingale under P^{**} . For any $\hat{\phi} \in \mathcal{P}$,

$$\begin{aligned} E^{P^{**}} \left[\sum_{t=1}^T \hat{\phi}_t \cdot \Delta \hat{S}_t^* \right] &= E^{P^*} \left[\sum_{t=1}^T \hat{\phi}_t \cdot \Delta \hat{S}_t^* \right] \\ &\quad + \frac{1}{2\|\tilde{Z}\|_\infty} E^{P^*} \left[\tilde{Z} \sum_{t=1}^T \hat{\phi}_t \cdot \Delta \hat{S}_t^* \right]. \end{aligned}$$

The first term on the right side of the equality above is zero, by Lemma 3.2.6, since S^* is a martingale under P^* . The second term there is zero by (3.38), since $Y = \sum_{t=1}^T \hat{\phi}_t \cdot \Delta \hat{S}_t^* \in L$. On setting $\hat{\phi}_t^j = 0$ for all $j \neq i$ and $t = 1, 2, \dots, T$, and applying Lemma 3.2.6 again, it follows that $S^{*,i}$ is a P^{**} -martingale for $i = 1, \dots, d$; and since this is trivially so for $i = 0$, it follows that S^* is a P^{**} martingale and hence P^{**} is an equivalent martingale measure that is different from P^* . $\square$

The following can be viewed as a form of the martingale representation theorem in a finite market model context. It gives an alternative characterization of completeness in a viable finite market model.

Theorem 3.3.3. *Suppose the finite market model is viable and P^* is an equivalent martingale measure. Then, the model is complete if and only if each real-valued martingale $M = \{M_t, \mathcal{F}_t, t = 0, 1, \dots, T\}$ under P^* has a representation of the form*

$$M_t = M_0 + \sum_{s=1}^t \hat{\phi}_s \cdot \Delta \hat{S}_s^*, \quad t = 0, 1, \dots, T, \quad (3.40)$$

for some $\hat{\phi} = (\hat{\phi}^1, \dots, \hat{\phi}^d)$ where for each $i = 1, \dots, d$, $\hat{\phi}^i = \{\hat{\phi}_t^i, t = 1, \dots, T\}$ and $\hat{\phi}_t^i$ is a real-valued $\mathcal{F}_{t-1}$ -measurable random variable for each t . (Here $\hat{S}^* = (S^{*,1}, S^{*,2}, \dots, S^{*,d})$.)

Proof. Suppose the model is complete and let $M = \{M_t, \mathcal{F}_t, t = 0, 1, \dots, T\}$ be a martingale under P^* . Then, $X = M_T S_T^0$ is a European contingent claim. Since the model is complete, there exists a replicating strategy ϕ for X . Then, $V_T^*(\phi) = X^* = M_T$, and by Theorem 3.3.1 we have for $t = 0, 1, \dots, T-1$,

$$V_t^*(\phi) = E^{P^*}[X^* | \mathcal{F}_t] = E^{P^*}[M_T | \mathcal{F}_t]. \quad (3.41)$$

Since M is a P^* -martingale, the last member above is equal to M_t P^* -a.s., and so it follows upon using (3.11)–(3.12) that for $t = 0, 1, \dots, T$,

$$\begin{aligned} M_t = V_t^*(\phi) &= V_0^*(\phi) + G_t^*(\phi) \\ &= V_0^*(\phi) + \sum_{s=1}^t \phi_s \cdot \Delta S_s^* \\ &= M_0 + \sum_{s=1}^t \hat{\phi}_s \cdot \Delta \hat{S}_s^* \end{aligned}$$

where $\hat{\phi} = \{(\phi_t^1, \dots, \phi_t^d), t = 1, \dots, T\}$ and $\phi_t^i \in \mathcal{F}_{t-1}$ for $t = 1, \dots, T$, $i = 1, \dots, d$. Here we have used the fact that $G_t^*(\phi)$ involves only the holdings in the risky assets.

Conversely, suppose the representation property holds. To show that the model is complete, consider a European contingent claim X . Define

$$M_t = E^{P^*}[X^* | \mathcal{F}_t], \quad t = 0, 1, \dots, T. \quad (3.42)$$

Then $M = \{M_t, \mathcal{F}_t, t = 0, 1, \dots, T\}$ is a martingale under P^* . Let $\hat{\phi}$ be as in the representation (3.40). By Lemma 3.2.5, $\hat{\phi}$ can be extended to a

trading strategy $\phi = (\phi^0, \hat{\phi}^1, \dots, \hat{\phi}^d)$ with initial value $M_0 S_0^0$. Then, for $t = 0, 1, \dots, T$,

$$\begin{aligned} V_t^*(\phi) &= V_0^*(\phi) + G_t^*(\phi) \\ &= M_0 + \sum_{s=1}^t \hat{\phi}_s \cdot \Delta \hat{S}_s^* \\ &= M_t. \end{aligned}$$

Hence, $V_T^*(\phi) = M_T = X^*$, and it follows that $V_T(\phi) = X$. Thus, ϕ is a replicating strategy for X . Since X was an arbitrary European contingent claim, it follows that the market is complete. $\square$

3.4. Pricing European Contingent Claims

In this section, we assume that the finite market model is viable and complete. Let P^* be the unique equivalent martingale measure. Consider a European contingent claim with value X at time T . Here we generalize from the situation treated in Chapter 2, where trading of the European contingent claim was allowed only at time zero, to allow it to be traded at all times $t = 0, 1, 2, \dots, T-1$. (We leave it as an exercise for the reader to formulate the appropriate notion of arbitrage and to verify that the unique arbitrage free initial price remains valid when the European contingent claim can be bought or sold only at time zero.)

To determine the arbitrage free price process for the European contingent claim, we consider a market that allows trading in the stock, bond and European contingent claim at each time $t = 0, 1, \dots, T-1$. Let $\{C_t, t = 0, 1, \dots, T\}$ be an $\{\mathcal{F}_t\}$ -adapted process, where C_t represents the price of the European contingent claim at time $t = 0, 1, \dots, T-1$ and $C_T = X$. A *trading strategy* in the stocks, bond and the European contingent claim is a collection $\psi = \{(\phi_t, \gamma_t), t = 1, \dots, T\}$ where for each t , $\phi_t = (\phi_t^0, \phi_t^1, \dots, \phi_t^d)$ is a $(d+1)$ -dimensional $\mathcal{F}_{t-1}$ -measurable random vector such that for $i = 0, 1, \dots, d$, ϕ_t^i represents the number of “shares” of asset i held over the time interval $(t-1, t]$, and γ_t is a real-valued $\mathcal{F}_{t-1}$ -measurable random variable representing the number of European contingent claims held over the time interval $(t-1, t]$. This trading strategy must be self-financing; i.e., its initial value is

$$V_0(\psi) = \phi_1 \cdot S_0 + \gamma_1 C_0,$$

and at each time $t = 1, \dots, T-1$,

$$\phi_t \cdot S_t + \gamma_t C_t = \phi_{t+1} \cdot S_t + \gamma_{t+1} C_t.$$

The value of the stocks-bond-contingent claim portfolio at time T is

$$V_T(\psi) = \phi_T \cdot S_T + \gamma_T X.$$

An *arbitrage opportunity* in the stocks-bond-contingent claim market is a trading strategy ψ such that $V_0(\psi) = 0$, $V_T(\psi) \geq 0$ and $E[V_T(\psi)] > 0$.

Remark. One may think of the stocks-bond-contingent claim market as a finite market model in its own right, with $d + 1$ risky assets (d stocks and one contingent claim) and one riskless asset (one bond). Then the above notions of trading strategy and arbitrage coincide with those defined earlier in this chapter.

Theorem 3.4.1. *Suppose the finite market model is viable and complete and P^* is the unique equivalent martingale measure. Then for any European contingent claim X ,*

$$\{S_t^0 E^{P^*}[X^* \mid \mathcal{F}_t], t = 0, 1, \dots, T\}$$

is the (unique) arbitrage free price process for the European contingent claim, where $X^ = X/S_T^0$ is the discounted value of X at time T .*

Remark. Here uniqueness of the price process is uniqueness up to indistinguishability of stochastic processes.

Proof. Let ϕ be a replicating strategy for X . Then by Theorem 3.3.1,

$$V_t(\phi) = S_t^0 E^{P^*}[X^* \mid \mathcal{F}_t], \quad t = 0, 1, \dots, T.$$

Let $\{C_t, t = 0, 1, \dots, T\}$ be the price process for the European contingent claim, where $C_T = X$.

We first show that if $P(C_s \neq V_s(\phi)) > 0$ for some s , then there is an arbitrage opportunity. Note $C_T = V_T(\phi) = X$. Suppose there is $s \in \{0, 1, \dots, T-1\}$ such that $P(C_s > V_s(\phi)) > 0$. Let $A = \{\omega : C_s(\omega) > V_s(\phi)(\omega)\}$. Then $A \in \mathcal{F}_s$. An investor could act as follows to achieve an arbitrage. The investor invests nothing in stocks, bond or contingent claim up to time s . If $C_s \leq V_s(\phi)$, the investor continues to invest nothing from time s to T . If $C_s > V_s(\phi)$, at time s , the investor sells one European contingent claim, invests $V_s(\phi)$ of the proceeds in the market from time s onwards according to the strategy $(\phi_{s+1}, \dots, \phi_T)$, and puts the remainder, $C_s - V_s(\phi)$, in the bond from time s to T . This strategy may be formally written as

$$\psi_t = \begin{cases} 0 & \text{for } t \leq s \\ \left(\phi_t^0 + \frac{C_s - V_s(\phi)}{S_s^0}, \phi_t^1, \dots, \phi_t^d, -1 \right) 1_A & \text{for } t > s. \end{cases}$$

It is readily verified to be self-financing. In particular, $V_t(\psi) = 0$ for $t \leq s$, and

$$1_A(\phi_{s+1} \cdot S_s + C_s - V_s(\phi) - C_s) = 0.$$

Now,

$$\begin{aligned} V_T(\psi) &= 1_A \left(\phi_T \cdot S_T + (C_s - V_s(\phi)) \frac{S_T^0}{S_s^0} - X \right) \\ &= 1_A \left((C_s - V_s(\phi)) \frac{S_T^0}{S_s^0} \right), \end{aligned}$$

since $V_T(\phi) = \phi_T \cdot S_T = X$, by the replicating property of ϕ . Thus, $V_T(\psi) \geq 0$ and $P(V_T(\psi) > 0) = P(A) > 0$, and ψ is an arbitrage.

Similarly, if $s \in \{0, 1, \dots, T-1\}$ such that $P(C_s < V_s(\phi)) > 0$ and $B = \{C_s < V_s(\phi)\}$, and we let

$$\psi_t = \begin{cases} 0 & \text{for } t \leq s \\ \left(-\phi_t^0 + \frac{V_s(\phi) - C_s}{S_s^0}, -\phi_t^1, \dots, -\phi_t^d, 1 \right) 1_B & \text{for } t > s, \end{cases}$$

for $t = 1, \dots, T$, then ψ represents an arbitrage. Thus we have shown that if $P(C_s \neq V_s(\phi) \text{ for some } s) > 0$, then there is an arbitrage opportunity.

It remains to show that if $C_s = V_s(\phi)$ for all s , there is no arbitrage opportunity in the stocks-bond-contingent claim market. For a contradiction, suppose $C_s = V_s(\phi)$ for all s and that $\psi = \{(\tilde{\phi}_t, \tilde{\gamma}_t), t = 1, \dots, T\}$ is an arbitrage opportunity. Then $V_0(\psi) = 0$, $V_T(\psi) \geq 0$ and $E[V_T(\psi)] > 0$. Let $V_t^*(\psi) = V_t(\psi)/S_t^0$, $t = 0, 1, \dots, T$. Then for $t \in \{1, \dots, T\}$,

$$\begin{aligned} E^{P^*}[V_t^*(\psi) \mid \mathcal{F}_{t-1}] &= E^{P^*}[\tilde{\phi}_t \cdot S_t^* + \tilde{\gamma}_t V_t^*(\phi) \mid \mathcal{F}_{t-1}] \\ &= \tilde{\phi}_t \cdot E^{P^*}[S_t^* \mid \mathcal{F}_{t-1}] + \tilde{\gamma}_t E^{P^*}[V_t^*(\phi) \mid \mathcal{F}_{t-1}] \\ &= \tilde{\phi}_t \cdot S_{t-1}^* + \tilde{\gamma}_t V_{t-1}^*(\phi) \end{aligned}$$

where for the last equality we have used the facts that S^* and $V^*(\phi)$ are martingales under P^* (cf. (3.29)). Using the self-financing property of ψ , we recognize the last expression above as $V_{t-1}^*(\psi)$. Hence, $\{V_t^*(\psi), \mathcal{F}_t, t = 0, 1, \dots, T\}$ is a P^* -martingale and so

$$E^{P^*}[V_T^*(\psi)] = E^{P^*}[V_0^*(\psi)]. \quad (3.43)$$

Now, $V_0^*(\psi) = V_0(\psi) = 0$, $V_T^*(\psi) \geq 0$, so it follows that $P^*(V_T^*(\psi) = 0) = 1$; and since P^* is equivalent to P , $P(V_T^*(\psi) = 0) = 1$. Hence, $P(V_T(\psi) = 0) = 1$, which contradicts the assumption that ψ is an arbitrage.

Thus, $C_t = V_t(\phi) = S_t^0 E^{P^*}[X^* \mid \mathcal{F}_t]$, $t = 0, 1, \dots, T$, defines the unique arbitrage free price process. $\square$

Remark. Examination of the above proof shows that the conclusion of the theorem still holds if the finite market model is viable and X is simply replicable; i.e., one does not need to assume completeness of the market model.

3.5. Incomplete Markets

In this section we give a brief introduction to incomplete markets in the finite market model setting. For simplicity, we consider only a single time period; i.e., we assume that $T = 1$. However, many of the properties described here generalize to multiperiod models. We assume that the finite market model has no arbitrage opportunities; i.e., it is viable. It follows that there is at least one equivalent martingale measure.

Let X be a European contingent claim. A *superhedging strategy* for X is a trading strategy ϕ such that $V_T(\phi) \geq X$. By Theorem 3.3.1, the initial value of such a strategy is given by $E^{P^*}[Y^*]S_0^0$ where $Y^* = V_T(\phi)/S_T^0$ and P^* is any equivalent martingale measure. Thus,

$$\begin{aligned} V_+(X) &\equiv \inf\{V_0(\phi) : \phi \text{ is a superhedging strategy for } X\} \\ &= \inf\{E^{P^*}[Y^*]S_0^0 : Y \text{ is replicable and } Y \geq X\}, \end{aligned} \quad (3.44)$$

for any equivalent martingale measure P^* . From the last line it follows that $V_+(X) \geq E^{P^*}[X^*]S_0^0$ for each equivalent martingale measure P^* . The infimum in the definition of $V_+(X)$ is actually attained by some superhedging strategy ϕ . To see this, recall that Ω is a finite set and note that $V_+(X)$ is the optimal value of the linear program

$$\begin{aligned} &\text{minimize} && \phi_1 \cdot S_0 \\ &\text{subject to} && \phi_1 \cdot S_1 \geq X, \\ &&& \phi_1 \in \mathbb{R}^{d+1}. \end{aligned}$$

(Here we have used the fact that $T = 1$. In particular, $\phi_1 \in \mathcal{F}_0$ is a constant vector.) Since the feasible set for this linear program is easily seen to be non-empty and the objective function is bounded below (by $E^{P^*}[X^*]S_0^0$ where P^* is any equivalent martingale measure), it follows that the optimal value is attained (cf. Bertsimas and Tsitsiklis [3], page 67). A superhedging strategy for X whose initial value equals $V_+(X)$ is called a *minimal superhedging strategy*.

If the contingent claim is initially priced at $V_+(X)$, a seller of the claim can invest the proceeds of the sale in a minimal superhedging strategy ϕ and the seller is protected from the risk of having sold the contingent claim, since the seller's portfolio value at time T will be $V_T(\phi) - X \geq 0$. Furthermore, if the contingent claim is initially priced at more than $V_+(X)$, there is an arbitrage opportunity for the seller (the proof is left to the reader).

On the other hand, a buyer of the contingent claim wants the initial price C_0 of the contingent claim to be such that the buyer can invest $-C_0$ in a trading strategy ϕ with $V_0(\phi) = -C_0$ and such that the buyer is protected from the risk of having bought the contingent claim; i.e., $V_T(\phi) + X \geq 0$.

The supremum of such initial prices C_0 is

$$\begin{aligned} V_-(X) &\equiv \sup\{-V_0(\phi) : \phi \text{ is a trading strategy, } V_T(\phi) + X \geq 0\} \\ &= \sup\{V_0(\phi) : \phi \text{ is a trading strategy, } V_T(\phi) \leq X\} \\ &= \sup\{E^{P^*}[Y^*]S_0^0 : Y \text{ is replicable and } Y \leq X\}, \end{aligned}$$

for any equivalent martingale measure P^* . In the next to last line above we have used the fact that ϕ is a trading strategy if and only if $-\phi$ is a trading strategy and that the value of a trading strategy at each time t is a linear function of the trading strategy. In a similar manner to that for $V_+(X)$, one can see that the supremum in the definition of $V_-(X)$ is attained and $V_-(X) \leq E^{P^*}[X^*]S_0^0$ for all equivalent martingale measures P^* . Moreover, if the initial price of the contingent claim is priced less than $V_-(X)$, there is an arbitrage opportunity for the buyer of the contingent claim.

Let $\mathcal{M}$ denote the set of equivalent martingale measures. Then, from the above, we have

$$V_-(X) \leq \inf_{P^* \in \mathcal{M}} E^{P^*}[X^*]S_0^0 \leq \sup_{P^* \in \mathcal{M}} E^{P^*}[X^*]S_0^0 \leq V_+(X). \quad (3.45)$$

If X is replicable, then from the Remark at the end of Section 3.4, we have that $V_+(X) = V_-(X)$ is the unique arbitrage free initial price for the contingent claim X .

If X is not replicable, there are no arbitrage free initial prices outside the closed interval $[V_-(X), V_+(X)]$. In fact, $V_-(X)$ and $V_+(X)$ are also not arbitrage free initial prices in this case. For $V_+(X)$, this follows from the fact that the inf in the definition of $V_+(X)$ is attained and a minimal superhedging strategy ϕ must have $V_T(\phi) > X$ with positive probability, since otherwise ϕ would be a replicating strategy for X . A similar argument applies for $V_-(X)$. Using the definition of $V_-(X)$, $V_+(X)$, and the martingale property of S^* under any equivalent martingale measure, one can use an argument by contradiction to show that any initial price in the open interval $(V_-(X), V_+(X))$ is arbitrage free. This last interval is non-empty if X is not replicable, as the following lemma shows.

Lemma 3.5.1. *Suppose that a European contingent claim X is not replicable. Then $V_-(X) < V_+(X)$.*

Proof. Let P^* be an equivalent martingale measure. Recall from the proof of the second fundamental theorem of asset pricing, Theorem 3.3.2, that if X is not replicable, then X^* is not in

$$L = \{V_T^*(\phi) : \phi \text{ is a trading strategy}\}.$$

Decomposing X^* into its component X_L^* in L and its non-trivial component Z that is perpendicular to L , we have

$$X^* = X_L^* + Z. \quad (3.46)$$

From the proof of Theorem 3.3.2 we see that P^{**} , defined by

$$P^{**}(\{\omega\}) = \left(1 + \frac{\tilde{Z}(\omega)}{2\|\tilde{Z}\|_\infty}\right) P^*(\{\omega\}), \quad \omega \in \Omega, \quad (3.47)$$

where $\tilde{Z}$ is given by (3.37), is an equivalent martingale measure. From Theorem 3.3.1, since $X_L^* = V_T^*(\phi)$ for some trading strategy ϕ , we have $E^{P^*}[X_L^*] = E^{P^{**}}[X_L^*] = V_0^*(\phi)$. Now,

$$E^{P^{**}}[Z] = E^{P^*}[Z] + \frac{Z \cdot Z}{2\|\tilde{Z}\|_\infty} > E^{P^*}[Z], \quad (3.48)$$

since $Z \neq 0$. Hence, $E^{P^*}[X^*] < E^{P^{**}}[X^*]$, and so using (3.45) we have

$$V_-(X) \leq E^{P^*}[X^*]S_0^0 < E^{P^{**}}[X^*]S_0^0 \leq V_+(X). \quad (3.49)$$

□

The following is a consequence of Theorem 3.3.1 and the proof of the above theorem.

Corollary 3.5.2. *A European contingent claim X is replicable if and only if $E^{P^*}[X^*]$ has the same value for all equivalent martingale measures P^* .*

We also have the following alternative characterization of $V_+(X)$ and $V_-(X)$, which implies that the first and last inequalities in (3.45) are actually equalities.

Theorem 3.5.3. *For any European contingent claim X ,*

$$\begin{aligned} V_+(X) &= \sup \left\{ E^{P^*}[X^*]S_0^0 : P^* \in \mathcal{M} \right\}, \\ V_-(X) &= \inf \left\{ E^{P^*}[X^*]S_0^0 : P^* \in \mathcal{M} \right\}. \end{aligned}$$

Proof. We prove the equality for $V_+(X)$; the proof for $V_-(X)$ is similar. Viewing probabilities on the finite set Ω as vectors in n -dimensional Euclidean space (where $n = |\Omega|$), let $P_1^*, \dots, P_J^*$ be a maximal linearly independent subset of $\mathcal{M}$. Then any element of $\mathcal{M}$ can be written in the form $\sum_{j=1}^J \theta_j P_j^*$ for some real constants θ_j , $j = 1, \dots, J$ satisfying $\sum_{j=1}^J \theta_j = 1$. It follows from Corollary 3.5.2 that a European contingent claim Y is replicable if and only if $E^{P^*}[Y^*]$ has the same value for all $P^* \in \mathcal{M}$, and this

occurs if and only if $E^{P_j^*}[Y^*]$ has the same value for $j = 1, \dots, J$. Then, by (3.44), $V_+(X)$ is the optimal value of the linear program

$$\begin{aligned} & \text{minimize} && \lambda \\ & \text{subject to} && \lambda = E^{P_j^*}[Y^*]S_0^0, \quad j = 1, \dots, J, \\ & && Y^* \geq X^*. \end{aligned}$$

The dual to this linear program is

$$\begin{aligned} & \text{maximize} && \psi \cdot X^* \\ & \text{subject to} && \psi = \sum_{j=1}^J \theta_j P_j^* S_0^0, \\ & && \psi \geq 0, \quad \sum_{j=1}^J \theta_j = 1, \\ & && \psi \in \mathbb{R}^n, \quad \theta \in \mathbb{R}^J. \end{aligned}$$

Let $\tilde{\mathcal{M}}$ denote the set of probabilities on Ω under which S^* is a martingale, and let

$$\mathcal{N} = \{\psi/S_0^0 : \psi \in \mathbb{R}^n \text{ is feasible for the above dual program}\}. \quad (3.50)$$

We claim that

$$\mathcal{N} = \tilde{\mathcal{M}} = \overline{\mathcal{M}}, \quad (3.51)$$

where $\overline{\mathcal{M}}$ denotes the closure of $\mathcal{M}$ in $\mathbb{R}^n$. This claim can be verified by noting the following.

- (a) Each element of $\mathcal{M}$ when multiplied by S_0^0 yields a feasible ψ for the dual program, and the set of feasible ψ for the dual program is closed, being the linear projection of the closed set of feasible pairs (ψ, θ) for the dual program (cf. Bertsimas and Tsitsiklis [3], Corollary 2.4, p. 74). Hence, $\overline{\mathcal{M}}$ is contained in $\mathcal{N}$.
- (b) For a feasible ψ for the dual program, ψ/S_0^0 is an element of $\tilde{\mathcal{M}}$. Although an element of $\tilde{\mathcal{M}}$ need not be strictly positive on all of Ω , it can be approximated arbitrarily closely by an element of $\mathcal{M}$, since there is at least one equivalent martingale measure. Hence, $\mathcal{N} \subset \tilde{\mathcal{M}} \subset \overline{\mathcal{M}}$.

Since the original linear program has a finite optimal value, the same is true of the dual program, and the optimal values for the two programs are the same and they are attained.

Combining the above, we obtain

$$V_+(X) = \max \left\{ E^Q[X^*]S_0^0 : Q \in \tilde{\mathcal{M}} \right\} = \sup \left\{ E^{P^*}[X^*]S_0^0 : P^* \in \mathcal{M} \right\}.$$

□

Now, by (3.45), Lemma 3.5.1 and Theorem 3.5.3, when X is not replicable,

$$V_-(X) = \inf_{P^* \in \mathcal{M}} E^{P^*}[X^*]S_0^0 < \sup_{P^* \in \mathcal{M}} E^{P^*}[X^*]S_0^0 = V_+(X), \quad (3.52)$$

and all of the prices in the non-empty interval $(V_-(X), V_+(X))$ are arbitrage free initial prices, and neither the buyer nor the seller can completely eliminate the risk of buying or selling the contingent claim for such a price. To further refine the choice of an initial arbitrage free price and a hedging strategy, various procedures have been proposed. For a summary of an approach based on utility maximization under the subjective probability P , see Davis [13].

For further discussion of incomplete markets in discrete time, see Föllmer and Schied [16] (Chapter 10) and Musiela and Rutkowski [32] (Chapter 4), for a start.

3.6. Separating Hyperplane Theorem

Theorem 3.6.1. (*Separating Hyperplane Theorem*) *Let F be a compact, convex, non-empty subset of $\mathbb{R}^n$. Let L be a non-empty linear subspace of $\mathbb{R}^n$. Suppose $F \cap L = \emptyset$. Then there exists a hyperplane*

$$H = \{x \in \mathbb{R}^n : z \cdot x = 0\} \text{ for some } z \in \mathbb{R}^n \setminus \{0\}$$

such that $L \subset H$ and $z \cdot x > 0$ for all $x \in F$.

Proof. Define $G = F - L = \{x \in \mathbb{R}^n : x = f - \ell \text{ for some } f \in F, \ell \in L\}$. Then G is convex, closed and non-empty. The convexity follows easily from that for F and L . To see that G is closed, consider a sequence $\{x_m = f_m - \ell_m\}_{m=1}^\infty$ in G where $f_m \in F$, $\ell_m \in L$ for all m . Suppose $x_m \rightarrow x \in \mathbb{R}^n$ as $m \rightarrow \infty$. Since F is compact, there is a subsequence $\{f_{m_r}\}_{r=1}^\infty$ converging to some $f \in F$. Then $\ell_{m_r} = -x_{m_r} + f_{m_r}$ converges to $-x + f$. Since L is closed we must have $-x + f \in L$. Hence $x = f - (-x + f) \in G$. Clearly G is non-empty, since F and L are both non-empty. Note that G does not contain the origin (otherwise F would intersect L non-trivially).

Let B denote the closed ball centered at the origin of radius $r > 0$ (where distance is measured by the Euclidean norm). Choose $r > 0$ such that $B \cap G \neq \emptyset$. Then $B \cap G$ is closed, bounded, non-empty and hence compact. So the continuous function $g(x) = \|x\|$ attains its infimum on $B \cap G$ at some $z \in B \cap G$, where $\|x\| = (x \cdot x)^{\frac{1}{2}}$ denotes the Euclidean norm of x . Note that $\|z\| \leq r$, since $z \in B$, and $z \neq 0$ since $0 \notin G$.

Now $\|x\| > r$ for $x \in G \setminus B$ and so combining this with the above we have $\|x\| \geq \|z\|$ for all $x \in G$. Then for any $\lambda \in (0, 1)$ and $x \in G$, $\lambda x + (1-\lambda)z \in G$

by the convexity of G and so

$$\|\lambda x + (1 - \lambda)z\|^2 \geq \|z\|^2 \text{ for all } x \in G, \lambda \in (0, 1).$$

Expanding and dividing through by λ yields:

$$2(1 - \lambda)x \cdot z - 2z \cdot z + \lambda(x \cdot x + z \cdot z) \geq 0.$$

Letting $\lambda \rightarrow 0$, we obtain

$$x \cdot z \geq z \cdot z \quad \text{for all } x \in G.$$

Then, $(f - \ell) \cdot z \geq z \cdot z$ for all $f \in F, \ell \in L$, which implies

$$f \cdot z \geq \ell \cdot z + z \cdot z \quad \text{for all } f \in F, \ell \in L.$$

Fix $f \in F$. Then

$$\ell \cdot z \leq f \cdot z - z \cdot z \quad \text{for all } \ell \in L.$$

But L is a linear space, so the above holds with $\gamma\ell$ in place of ℓ for all $\gamma \in \mathbb{R}$. The only way this can be true is if $\ell \cdot z = 0$ for all $\ell \in L$. Hence, $f \cdot z \geq z \cdot z > 0$ for all $f \in F$.

Let $H = \{x \in \mathbb{R}^n : x \cdot z = 0\}$. Then from the above, $L \subset H$ and $f \cdot z > 0$ for all $f \in F$. $\square$

3.7. Exercises

1. Consider the multi-period CRR binomial model introduced in Chapter 2. Assuming $u > d > 0$, verify that this model is viable if and only if $d < 1 + r < u$. In this case, verify that the model is complete.

2. Let $T = 2$, $\Omega = \{\omega_1, \dots, \omega_4\}$, $P(\{\omega_i\}) > 0$ for $i = 1, \dots, 4$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \emptyset, \Omega\}$, and $\mathcal{F}_2$ be the collection of all subsets of Ω . Consider a riskless asset with price process $S^0 = \{S_t^0, t = 0, 1, 2\}$ where $S_t^0 = 1$ for all t , and a risky asset with price process $S^1 = \{S_t^1, t = 0, 1, 2\}$ such that

$$\begin{aligned} S_0^1(\omega_1) &= 5, & S_1^1(\omega_1) &= 8, & S_2^1(\omega_1) &= 9 \\ S_0^1(\omega_2) &= 5, & S_1^1(\omega_2) &= 8, & S_2^1(\omega_2) &= 6 \\ S_0^1(\omega_3) &= 5, & S_1^1(\omega_3) &= 4, & S_2^1(\omega_3) &= 5 \\ S_0^1(\omega_4) &= 5, & S_1^1(\omega_4) &= 4, & S_2^1(\omega_4) &= 2. \end{aligned}$$

Then

$$X = \max(0, S_0^1 - 6, S_1^1 - 6, S_2^1 - 6)$$

is the value at time T of a so-called *look-back* option, where this value depends on the maximum of the prices of the underlying asset S^1 over the time interval from 0 to T .

- (a) Draw a tree to indicate the possible “paths” followed by the risky asset price process S^1 .
- (b) Find an equivalent martingale measure for the model.
- (c) Find a replicating strategy for the option whose value at time T is given by X .
- (d) What is the arbitrage free price for the option at time zero?

3. Consider a finite market model with $T = 2$, $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$, and $P(\{\omega_i\}) > 0$ for $i = 1, \dots, 5$. Suppose there are two assets, a riskless asset with price process $S^0 = \{S_t^0, t = 0, 1, 2\}$ where $S_t^0 = (1 + r)^t$ for $t = 0, 1, 2$, and some $r \geq 0$, and a risky asset with price process $S^1 = \{S_t^1, t = 0, 1, 2\}$ where

$$\begin{aligned} S_0^1(\omega_1) &= 5, & S_1^1(\omega_1) &= 8, & S_2^1(\omega_1) &= 9 \\ S_0^1(\omega_2) &= 5, & S_1^1(\omega_2) &= 8, & S_2^1(\omega_2) &= 7 \\ S_0^1(\omega_3) &= 5, & S_1^1(\omega_3) &= 4, & S_2^1(\omega_3) &= 6 \\ S_0^1(\omega_4) &= 5, & S_1^1(\omega_4) &= 4, & S_2^1(\omega_4) &= 5 \\ S_0^1(\omega_5) &= 5, & S_1^1(\omega_5) &= 4, & S_2^1(\omega_5) &= 2. \end{aligned}$$

Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \sigma\{S_0^1, S_1^1\}$ and $\mathcal{F}_2 = \sigma\{S_0^1, S_1^1, S_2^1\}$.

- (a) Draw a tree to indicate the possible “paths” followed by the risky asset price process S^1 .
 - (b) Suppose $r = 0.1$. Is there an equivalent martingale measure for this model? If there is one, is it unique? If there is not one, demonstrate an arbitrage opportunity. What are the answers to the last three questions if $r = 1$?
- 4.** Let $T = 1$, $S_t^0 = (1 + r)^t$ where $r = \frac{1}{3}$, $t = 0, 1$, $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, and $P(\{\omega_i\}) > 0$ for $i = 1, 2, 3, 4$. Suppose that there are two risky assets with price processes S^1, S^2 :

i	S_0^i	S_1^i			
		ω_1	ω_2	ω_3	ω_4
1	2	4	4	$8/3$	$4/3$
2	5	8	$16/3$	$16/3$	8

- (a) Specify the space

$$L = \{G_T^*(\phi) : \phi \text{ is a trading strategy with } V_0(\phi) = 0\}$$
 for this example.
- (b) Are there any arbitrage opportunities for this example? If so, find all of them.

- (c) Are there any equivalent martingale measures for this example? If so, find all of them.
- 5.** Show that the separating hyperplane theorem follows from the Hahn-Banach theorem.