

Binomial Model

In this chapter we consider a simple discrete (primary) financial market model called the binomial or Cox-Ross-Rubinstein (CRR) [12] model. In the context of this model, we derive the unique arbitrage free prices for a European and an American contingent claim.

For this chapter and the next, we shall need various notions from probability and the theory of discrete time stochastic processes, including conditional expectations, martingales, supermartingales and stopping times. For the convenience of the reader, a summary of some relevant concepts and results is provided in Appendices A and B. For further details, the reader is encouraged to consult Chung [9] or D. Williams [38].

Regarding general notation used throughout this book, we note that if X is a real-valued random variable defined on a probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{G}$ is a sub- σ -algebra of $\mathcal{F}$, we shall sometimes use the notation $X \in \mathcal{G}$ to indicate that X is $\mathcal{G}$ -measurable. Also, two random variables will be considered equal if they are equal almost surely, and two stochastic processes will be considered equal if they are indistinguishable (see Appendices B and C). In this chapter and the next, we restrict attention to finite sample spaces and consider only probability measures that give positive probability to each individual outcome. Consequently, in these chapters, equality of random variables and indistinguishability of discrete time stochastic processes actually entail equality surely.

2.1. Binomial or CRR Model

The CRR model is a simple discrete time model for a financial market. There are finitely many times $t = 0, 1, 2, \dots, T$ (where $T < \infty$ is a positive

integer and successive times are successive integers). At each of these times the values of two assets can be observed. One asset is a risky security called a stock, and the other is a riskless security called a bond.

The *bond* is assumed to yield a constant rate of return $r \geq 0$ over each time period $(t-1, t]$; and so assuming the bond is valued at \$1 at time zero, the value of the bond at time t is given by

$$B_t = (1+r)^t, \quad t = 0, 1, \dots, T. \quad (2.1)$$

We measure holdings in the bond in units, where the value of one unit at time t is B_t .

The *stock* price process is modeled as an exponential random walk such that S_0 is a strictly positive constant and

$$S_t = S_{t-1}\xi_t, \quad t = 1, 2, \dots, T, \quad (2.2)$$

where $\{\xi_t, t = 1, 2, \dots, T\}$ is a sequence of independent and identically distributed random variables with

$$P(\xi_t = u) = p = 1 - P(\xi_t = d), \quad (2.3)$$

where $p \in (0, 1)$ and $0 < d < 1+r < u$. The last two conditions are assumed to avoid arbitrage opportunities in the primary market model and to ensure stock prices are random and strictly positive. Note that

$$S_t = S_0 \prod_{i=1}^t \xi_i, \quad t = 0, 1, \dots, T. \quad (2.4)$$

(Here we assume that an empty product is defined to take the value 1.) One may represent the possible paths that S_t follows using a binary tree (see Figure 1). Note that there are only three distinct values for S_2 ; i.e., the two middle dots in Figure 1 have the same value for S_2 . The points have been drawn as two distinct points to emphasize the fact that they may be reached by different paths, that is, through different values for the sequence S_0, S_1, S_2 . For pricing and hedging of some contingent claims under the binomial model (e.g., a European contingent claim whose payoff depends only on the terminal value of the stock price or an American contingent claim whose payoff at time t depends only on the stock price at that time), one may use a so-called recombining tree in which only the distinct values of S_i are indicated (in particular, the two middle dots associated with values of S_2 in Figure 1 are identified). However, in general one needs the full path structure of the stock price process to price and hedge contingent claims; cf. Exercise 3 at the end of this chapter.

For concreteness, and without loss of generality, we assume that the probability space $(\Omega, \mathcal{F}, P)$ on which our random variables are defined is such that Ω is the finite set of 2^T possible outcomes for the values of the

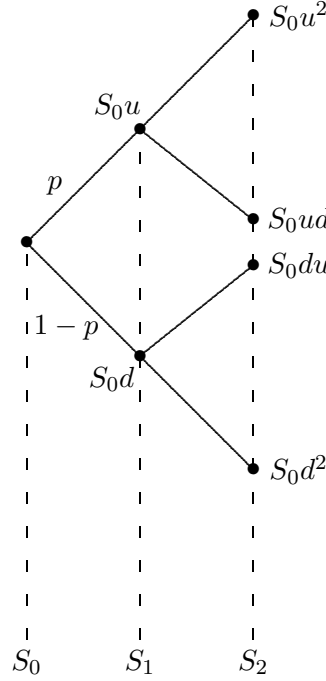


Figure 1. Binary tree for $T = 2$

stock price $(T + 1)$ -tuple, $(S_0, S_1, S_2, \dots, S_T)$; $\mathcal{F}$ is the σ -algebra consisting of all possible subsets of Ω ; and P is the probability measure on $(\Omega, \mathcal{F})$ associated with the Bernoulli probability p . Then, for example,

$$P((S_0, S_1, \dots, S_T) = (S_0, S_0u, S_0u^2, \dots, S_0u^T)) = p^T. \quad (2.5)$$

Indeed, each of the 2^T possible outcomes in Ω has strictly positive probability under P . The expectation operator under P will be denoted by $E[\cdot]$.

To describe the information available to the investor at time t , we introduce the σ -algebra generated by the stock prices up to and including time t ; i.e., let

$$\mathcal{F}_t = \sigma\{S_0, S_1, \dots, S_t\}, \quad t = 0, 1, \dots, T. \quad (2.6)$$

In particular, with our concrete probability space, $\mathcal{F}_T = \mathcal{F}$. The family $\{\mathcal{F}_t, t = 0, 1, \dots, T\}$ is called a *filtration*. We will often write it simply as $\{\mathcal{F}_t\}$.

A *trading strategy* (in the primary market) is a collection of pairs of random variables

$$\phi = \{(\alpha_t, \beta_t), t = 1, 2, \dots, T\} \quad (2.7)$$

where the random variable α_t represents the number of shares of stock to be held over the time interval $(t - 1, t]$ and the random variable β_t represents

the number of units of the bond to be held over the time interval $(t-1, t]$. For simplicity, we allow α_t, β_t to take any values in $\mathbb{R}$. In particular, we do not restrict to integer numbers of shares of stock or units of the bond, and we allow short selling of shares ($\alpha_t < 0$) and borrowing ($\beta_t < 0$). We think of trading occurring at time $t-1$ to determine the portfolio holdings (α_t, β_t) until the next trading time t . To avoid strategies that anticipate the future, it is assumed that α_t, β_t are $\mathcal{F}_{t-1}$ -measurable random variables for $t = 1, 2, \dots, T$. (In this discrete model setting, this simply means that α_t and β_t can be expressed as real-valued functions of $(S_0, S_1, \dots, S_{t-1})$. In fact, the dependence on S_0 is trivial since S_0 is assumed to be a constant.) Thus, the holdings in stock and bond over the time period $(t-1, t]$ can depend only on the stock prices observed up to and including time $t-1$. We will restrict attention here to *self-financing* trading strategies, namely, those trading strategies ϕ such that

$$\alpha_t S_t + \beta_t B_t = \alpha_{t+1} S_t + \beta_{t+1} B_t, \quad t = 1, 2, \dots, T-1, \quad (2.8)$$

and the investor's initial wealth is equal to

$$\alpha_1 S_0 + \beta_1 B_0. \quad (2.9)$$

We will simply refer to these as trading strategies rather than using the longer term “self-financing trading strategies”. We say that a trading strategy ϕ represents a portfolio whose *value* at time t is given by $V_t(\phi)$, where

$$V_0(\phi) = \alpha_1 S_0 + \beta_1 B_0, \quad (2.10)$$

$$V_t(\phi) = \alpha_t S_t + \beta_t B_t, \quad t = 1, 2, \dots, T. \quad (2.11)$$

An *arbitrage opportunity* (in the primary market) is a trading strategy ϕ such that $V_0(\phi) = 0$, $V_T(\phi) \geq 0$ and $E[V_T(\phi)] > 0$. We note that, in the presence of the preceding conditions, the last condition is equivalent to $P(V_T(\phi) > 0) > 0$.

2.2. Pricing a European Contingent Claim

A *European contingent claim* (ECC) is represented by an $\mathcal{F}_T$ -measurable random variable X . This is interpreted to mean that the value of the contingent claim at the expiration time T is given by X . For example, a *European call option* with strike price $K \in (0, \infty)$ and expiration date T is represented by $X = (S_T - K)^+ \equiv \max\{0, S_T - K\}$. The meaning of X in this case is as follows. If $S_T > K$, the holder of the European call option will exercise it at T and, after selling a share of stock just purchased for K , will make a profit of $S_T - K$ (ignoring whatever was paid for the option in the first place). Thus, in this case, the value of the option at T is $S_T - K$. On the other hand, if $S_T \leq K$, the holder of the option will not exercise it at T and

its value is 0 at T . Similarly, a European put option with the same strike price and expiration date is represented by $X = (K - S_T)^+$.

A *replicating (or hedging) strategy* for a European contingent claim X is a trading strategy ϕ such that $V_T(\phi) = X$. If there exists such a replicating strategy, the contingent claim is said to be *attainable* (or *redundant*).

In this section, we derive the (initial) arbitrage free price for a European contingent claim. Existence of such a price depends on the existence of a so-called risk neutral probability, and uniqueness depends on there being a replicating strategy for the contingent claim.

2.2.1. Single Period Case. We begin by examining the single period case where $T = 1$. For this case, we first show that there is a replicating strategy for any European contingent claim X . Given X , we seek a trading strategy $\phi = (\alpha_1, \beta_1)$, where α_1 and β_1 are constants such that

$$V_1(\phi) \equiv \alpha_1 S_1 + \beta_1 B_1 = X. \quad (2.12)$$

Now S_1 has two possible values, $S_0 u$, $S_0 d$, and X is a function of S_1 , since it is $\mathcal{F}_1 = \sigma\{S_0, S_1\}$ -measurable. Let X^u denote the value of X when $S_1 = S_0 u$ and X^d denote the value of X when $S_1 = S_0 d$. Then considering these two possible outcomes, (2.12) yields two equations for the two deterministic unknowns α_1, β_1 :

$$\alpha_1 S_0 u + \beta_1 (1 + r) = X^u \quad (2.13)$$

$$\alpha_1 S_0 d + \beta_1 (1 + r) = X^d. \quad (2.14)$$

Solving for α_1, β_1 yields

$$\alpha_1 = \frac{X^u - X^d}{(u - d)S_0}, \quad (2.15)$$

$$\beta_1 = \frac{1}{1 + r} \left(\frac{uX^d - dX^u}{u - d} \right). \quad (2.16)$$

The right member of (2.15) is sometimes written informally as $\frac{\delta X}{\delta S}$. The initial wealth needed to finance this strategy (sometimes called the manufacturing cost of the contingent claim) is

$$\begin{aligned} V_0 &= \alpha_1 S_0 + \beta_1 B_0 \\ &= \frac{1}{(1 + r)(u - d)} \left((1 + r - d)X^u + (u - (1 + r))X^d \right) \\ &= \frac{1}{1 + r} \left(p^* X^u + (1 - p^*) X^d \right) \\ &= E^{p^*}[X^*], \end{aligned} \quad (2.17)$$

where $X^* = X/(1+r)$, $p^* = \frac{1+r-d}{u-d}$, and $E^{p^*}[\cdot]$ denotes the expectation operator with p replaced by p^* . (For brevity, in this chapter, when probabilities or (conditional) expectations are those obtained by using p^* in place of p , we shall simply say that the probability or expectation is computed under p^* .)

Note that $p^* \in (0,1)$ and so, just as for p , each of the two possible outcomes for (S_0, S_1) has positive probability under p^* . Furthermore, $E^{p^*}[S_1] = (1+r)S_0$, and so the discounted stock price process

$$\left\{ S_0, \frac{1}{1+r} S_1 \right\}$$

is a martingale under p^* (relative to the filtration $\{\mathcal{F}_t\}$). Thus, under p^* , the average rate of return of the risky asset is the same as that of the riskless asset. For this reason p^* is called the *risk neutral probability*. (A person is said to be “risk averse” if the person prefers the expected value of a payoff to the random payoff itself. A person is said to be “risk preferring” if the person prefers the random payoff to the expected value of the payoff. A person is “risk neutral” if the person is neither risk averse nor risk preferring; in other words the person is indifferent, having no preference for the expected payoff versus the random payoff. The probability p^* is called a *risk neutral probability* because under p^* , a risk neutral investor would be indifferent to the choice at time zero of investing S_0 in stock or bond, since both investments have the same expected payoff at time 1 under p^* .)

It is important to realize that computing expectations under p^* is a mathematical device. We are not assuming that the stock price actually moves according to this probability. That is, p^* may be unrelated to the subjective probability p that we associate with the binomial model for movements in the stock price.

For the next theorem, we need the notion of an arbitrage opportunity in the market consisting of the stock, bond and contingent claim. For this, we suppose that the price of the contingent claim at time zero is C_0 , a constant. A *trading strategy* in stock, bond and the contingent claim is a triple $\psi = (\alpha_1, \beta_1, \gamma_1)$ of $\mathcal{F}_0$ -measurable random variables (these will actually be constants), where α_1 represents the number of shares of stock held over $(0, 1]$, β_1 represents the number of units of the bond held over $(0, 1]$, and γ_1 represents the number of units of the contingent claim held over $(0, 1]$. The initial value of the portfolio associated with ψ is $V_0(\psi) = \alpha_1 S_0 + \beta_1 B_0 + \gamma_1 C_0$. The value of this portfolio at time one is $V_1(\psi) = \alpha_1 S_1 + \beta_1 B_1 + \gamma_1 X$. An *arbitrage opportunity* in the stock-bond-contingent claim market is a trading strategy $\psi = (\alpha_1, \beta_1, \gamma_1)$ such that $V_0(\psi) = 0$, $V_1(\psi) \geq 0$ and $E[V_1(\psi)] > 0$.

Theorem 2.2.1. $V_0 = E^{p^*}[X^*]$ is the unique arbitrage free initial price for the European contingent claim X .

Proof. Let $\phi^* = (\alpha_1^*, \beta_1^*)$ denote the replicating strategy (in stock and bond) for the contingent claim X ; cf. (2.15)–(2.16).

First we show that if the initial price C_0 of the contingent claim is anything other than V_0 , then there is an arbitrage opportunity in the stock-bond-contingent claim market. Suppose $C_0 > V_0$. Then an investor starting with zero initial wealth could sell one contingent claim ($\gamma_1 = -1$) for C_0 , invest V_0 in the replicating strategy $\phi^* = (\alpha_1^*, \beta_1^*)$ and invest the remainder, $C_0 - V_0$, in bond. Thus, the trading strategy in stock, bond and contingent claim would be $(\alpha_1^*, \beta_1^* + C_0 - V_0, -1)$. This has an initial value of zero, and its value at time one is

$$\alpha_1^* S_1 + \beta_1^* B_1 + (C_0 - V_0) B_1 - X. \quad (2.18)$$

But the strategy (α_1^*, β_1^*) was chosen so that

$$\alpha_1^* S_1 + \beta_1^* B_1 = X, \quad (2.19)$$

and so it follows that the value at time one of the stock-bond-contingent claim portfolio is

$$(C_0 - V_0) B_1 > 0. \quad (2.20)$$

Thus, this represents an arbitrage opportunity. Similarly, if $C_0 < V_0$, then the investor can use the strategy $(-\alpha_1^*, -\beta_1^* + V_0 - C_0, 1)$ to create an arbitrage opportunity.

Now we show that if $C_0 = V_0$, then there is no arbitrage opportunity in the stock-bond-contingent claim market. Suppose that $\psi = (\alpha_1, \beta_1, \gamma_1)$ is a trading strategy in stock, bond and the contingent claim, with an initial value $V_0(\psi) = \alpha_1 S_0 + \beta_1 + \gamma_1 C_0$ of zero and non-negative value $V_1(\psi)$ at time one. The value of the portfolio at time one is

$$V_1(\psi) = \alpha_1 S_1 + \beta_1 B_1 + \gamma_1 X \quad (2.21)$$

and so

$$\begin{aligned} E^{p^*}[V_1(\psi)] &= \alpha_1 E^{p^*}[S_1] + \beta_1(1+r) + \gamma_1 E^{p^*}[X] \\ &= \alpha_1(1+r)S_0 + \beta_1(1+r) + \gamma_1(1+r)C_0 \\ &= (1+r)V_0(\psi) \\ &= 0, \end{aligned}$$

where we have used the fact that $\alpha_1, \beta_1, \gamma_1$ are constants (as they are $\mathcal{F}_0$ -measurable), plus the martingale property of the discounted stock price process under p^* and the assumption that $C_0 = V_0 = E^{p^*}[X]/(1+r)$. Now, since $V_1(\psi) \geq 0$ and the probability associated with p^* gives positive probability to all possible outcomes, it follows from the equality above that $V_1(\psi) = 0$, and hence $E[V_1(\psi)] = 0$. Thus, there cannot be an arbitrage opportunity. $\square$

Examination of the proof of Theorem 2.2.1 reveals that existence of an arbitrage free price for the contingent claim depends on the martingale property of $\{S_0, \frac{1}{1+r}S_1\}$ under p^* and uniqueness of the price depends on the existence of a replicating strategy for the contingent claim. These aspects will hold true more generally.

2.2.2. Multi-Period Case. We now consider the general binomial model where T is any fixed positive integer. In this subsection p^* has the same value as in the single period case, namely, $p^* = \frac{1+r-d}{u-d}$. Just as in the single period case, under the probability associated with p or p^* , all of the 2^T possible outcomes for the $(T+1)$ -tuple $(S_0, S_1, S_2, \dots, S_T)$ have positive probability.

We first show that there is a replicating strategy for a European contingent claim X . For this, given X , we seek a trading strategy $\phi = \{(\alpha_t, \beta_t), t = 1, \dots, T\}$ such that

$$V_T(\phi) \equiv \alpha_T S_T + \beta_T B_T = X. \quad (2.22)$$

This is developed by working backwards through the binary tree.

Let $V_T = X$. Since X is an $\mathcal{F}_T$ -measurable random variable and the $(T+1)$ -tuple $(S_0, S_1, \dots, S_T)$ can take only finitely many possible values, it follows that

$$V_T = f(S_0, S_1, \dots, S_T)$$

for some real-valued function f of $T+1$ variables. Firstly, suppose we condition on knowing the value of $S_0, S_1, \dots, S_{T-1}$. Then the cost and associated trading strategy for manufacturing the contingent claim over the time period $(T-1, T]$ can be computed in a very similar manner to that for the single period model. Given $S_0, S_1, \dots, S_{T-1}$, there are two possible values for V_T at time T , depending on whether $S_T = S_{T-1}u$ or $S_T = S_{T-1}d$. Denote these two values by V_T^u and V_T^d . In fact, $V_T^u = f(S_0, S_1, \dots, S_{T-1}, S_{T-1}u)$ and $V_T^d = f(S_0, S_1, \dots, S_{T-1}, S_{T-1}d)$. Note that these are $\mathcal{F}_{T-1}$ -measurable random variables (the notation V_T^u, V_T^d hides this fact but has the advantage that it makes the formulas for the replicating strategy appear simpler). If the contingent claim is a European call option with strike price K and expiration date T , then $X = (S_T - K)^+$ and $V_T^u = (S_{T-1}u - K)^+$, $V_T^d = (S_{T-1}d - K)^+$.

Now, for any European contingent claim X , by similar analysis to that for the single period case, to ensure that $V_T(\phi) = X$ we obtain the following allocations for the time period $(T-1, T]$:

$$\alpha_T = \frac{V_T^u - V_T^d}{(u-d)S_{T-1}} \quad (2.23)$$

$$\beta_T = \frac{1}{(1+r)^T} \left(\frac{uV_T^d - dV_T^u}{u-d} \right) \quad (2.24)$$

and the capital required at time $T - 1$ to finance these allocations in a self-financing manner is

$$\begin{aligned} V_{T-1} &= \frac{1}{1+r} \left(p^* V_T^u + (1-p^*) V_T^d \right) \\ &= \frac{1}{1+r} E^{p^*} [V_T | \mathcal{F}_{T-1}], \\ &= \frac{1}{1+r} E^{p^*} [X | \mathcal{F}_{T-1}], \end{aligned} \quad (2.25)$$

where $p^* = \frac{1+r-d}{u-d}$ and $E^{p^*}[\cdot | \mathcal{F}_{T-1}]$ denotes the conditional expectation, given $\mathcal{F}_{T-1}$, under p^* .

We can find a trading strategy $\phi = \{(\alpha_t, \beta_t), t = 1, 2, \dots, T\}$ with associated value process $\{V_t(\phi), t = 0, 1, \dots, T\}$ such that $V_T = X$ by proceeding inductively backwards through the binary tree as follows. For the induction step, fix $t \in \{1, \dots, T-1\}$ and assume that (self-financing) allocations $(\alpha_{t+1}, \beta_{t+1}), \dots, (\alpha_T, \beta_T)$ have been determined for the time periods $(t, t+1], \dots, (T-1, T]$ with values $V_t, \dots, V_{T-1}$, at times $t, t+1, \dots, T-1$, respectively, such that $V_s = \frac{1}{(1+r)^{T-s}} E^{p^*} [X | \mathcal{F}_s]$ for $s = t, t+1, \dots, T-1$, and the value of (α_T, β_T) at time T is $V_T = X$. Given $S_0, S_1, \dots, S_{t-1}$, the holdings α_t and β_t for the time period $(t-1, t]$ are chosen so that the value associated with these holdings at time t is the same as that of the random variable V_t ; i.e., letting V_t^u and V_t^d denote the two possible values of V_t given $S_0, S_1, \dots, S_{t-1}$, define

$$\alpha_t = \frac{V_t^u - V_t^d}{(u-d)S_{t-1}}, \quad (2.26)$$

$$\beta_t = \frac{1}{(1+r)^t} \left(\frac{uV_t^d - dV_t^u}{u-d} \right). \quad (2.27)$$

One can readily check, using the induction hypothesis, that the capital needed at time $t-1$ to finance these holdings in a self-financing manner is

$$\begin{aligned} V_{t-1} &= \frac{1}{1+r} E^{p^*} [V_t | \mathcal{F}_{t-1}] \\ &= \frac{1}{(1+r)^{T-t+1}} E^{p^*} \left[E^{p^*} [X | \mathcal{F}_t] | \mathcal{F}_{t-1} \right] \\ &= \frac{1}{(1+r)^{T-t+1}} E^{p^*} [X | \mathcal{F}_{t-1}]. \end{aligned}$$

Here we have used the tower property of conditional expectations; cf. Appendix A.

This completes the induction step and so it follows that the above procedure constructs a (self-financing) trading strategy, $\phi = \{(\alpha_t, \beta_t), t =$

$1, \dots, T\}$ with value process $\{V_t(\phi), t = 0, 1, \dots, T\}$ satisfying

$$V_t(\phi) = \frac{1}{(1+r)^{T-t}} E^{p^*}[X | \mathcal{F}_t] \quad (2.28)$$

for $t = 0, 1, \dots, T$. In particular, $V_T(\phi) = X$ and

$$V_0 = \frac{1}{(1+r)^T} E^{p^*}[X]. \quad (2.29)$$

Here we have used the fact that $\mathcal{F}_0$ is trivial, being generated by the constant value S_0 .

To prove that there exists an arbitrage free price for the contingent claim X , we need the following properties. For this, let $S_t^* = S_t/B_t = S_t/(1+r)^t$, $t = 0, 1, \dots, T$. The process $S^* = \{S_t^*, t = 0, 1, \dots, T\}$ is called the *discounted stock price process*.

Lemma 2.2.2. *The process $\{S_t^*, \mathcal{F}_t, t = 0, 1, \dots, T\}$ is a martingale under p^* .*

Proof. Clearly S_t^* is $\mathcal{F}_t$ -measurable, and S_t^* has finite mean for each t , since S_t^* takes only finitely many values. To verify the conditional expectation property, fix $t \in \{1, \dots, T\}$. Then, using the fact that $\mathcal{F}_{t-1}$ is generated by $S_0, S_1, \dots, S_{t-1}$ and ξ_t is independent of this σ -algebra, we have

$$\begin{aligned} E^{p^*}[S_t^* | \mathcal{F}_{t-1}] &= \frac{1}{(1+r)^t} E^{p^*}[S_{t-1}\xi_t | \mathcal{F}_{t-1}] \\ &= \frac{1}{(1+r)^t} S_{t-1} E^{p^*}[\xi_t] \\ &= \frac{S_{t-1}^*}{1+r} (p^*u + (1-p^*)d) \\ &= S_{t-1}^*, \end{aligned} \quad (2.30)$$

where we have used the definition of $p^* = \frac{1+r-d}{u-d}$ to obtain the last line. $\square$

Lemma 2.2.3. *Let ϕ be a trading strategy in stock and bond with value process $\{V_t(\phi), t = 1, \dots, T\}$. Consider the discounted value process $\{V_t^*(\phi) = V_t(\phi)/(1+r)^t, t = 0, 1, \dots, T\}$. Then $\{V_t^*(\phi), \mathcal{F}_t, t = 0, 1, \dots, T\}$ is a martingale under p^* .*

Proof. Let $\phi = \{(\alpha_t, \beta_t), t = 1, \dots, T\}$. Note that $V_t^*(\phi) = \alpha_t S_t^* + \beta_t$ and $\alpha_t, \beta_t \in \mathcal{F}_{t-1}$ for $t = 1, \dots, T$, and $V_0^*(\phi) = \alpha_1 S_0^* + \beta_1$. It is straightforward to show that $V_t^*(\phi)$ is $\mathcal{F}_t$ -measurable and integrable for $t = 0, 1, \dots, T$. Recall from Lemma 2.2.2 that $\{S_t^*, \mathcal{F}_t, t = 0, 1, \dots, T\}$ is a martingale under p^* . Therefore, for $t = 1, \dots, T$,

$$\begin{aligned} E^{p^*}[V_t^*(\phi) | \mathcal{F}_{t-1}] &= \alpha_t E^{p^*}[S_t^* | \mathcal{F}_{t-1}] + \beta_t \\ &= \alpha_t S_{t-1}^* + \beta_t. \end{aligned} \quad (2.31)$$

By factoring out $1/(1+r)^{t-1}$ and using the self-financing property of ϕ , it follows that for $t = 2, \dots, T$

$$\begin{aligned} E^{p^*} [V_t^*(\phi) \mid \mathcal{F}_{t-1}] &= \frac{1}{(1+r)^{t-1}} (\alpha_t S_{t-1} + \beta_t B_{t-1}) \\ &= \frac{1}{(1+r)^{t-1}} (\alpha_{t-1} S_{t-1} + \beta_{t-1} B_{t-1}) \\ &= V_{t-1}^*(\phi). \end{aligned}$$

For $t = 1$,

$$\begin{aligned} E^{p^*} [V_1^*(\phi) \mid \mathcal{F}_0] &= \alpha_1 S_0 + \beta_1 B_0 \\ &= V_0(\phi) = V_0^*(\phi), \end{aligned}$$

by definition. Hence the desired martingale property holds. $\square$

The following theorem is the multi-period analogue of the single period Theorem 2.2.1. Before proceeding to this, we must specify the notion of arbitrage in stock, bond and contingent claim to be used in the multi-period context. Let C_0 be the price charged for the contingent claim at time zero. Since we are specifying a price only for the contingent claim at time zero, trading in the contingent claim will be allowed only initially, whereas changes in the stock and bond holdings can occur at each of the times $t = 0, 1, \dots, T-1$. A *trading strategy* in stock, bond and the contingent claim is a collection $\psi = \{(\alpha_t, \beta_t), t = 1, 2, \dots, T; \gamma_1\}$ where for $t = 1, 2, \dots, T$, α_t, β_t are $\mathcal{F}_{t-1}$ -measurable random variables representing the holdings in stock and bond, respectively, to be held over the time interval $(t-1, t]$, and γ_1 is an $\mathcal{F}_0$ -measurable random variable (actually a constant) representing the number of units of the contingent claim to be held over the time interval $(0, T]$. The trading strategy must be self-financing; i.e., its initial value is

$$V_0(\psi) = \alpha_1 S_0 + \beta_1 B_0 + \gamma_1 C_0, \quad (2.32)$$

and at each time $t = 1, \dots, T-1$,

$$\alpha_t S_t + \beta_t B_t = \alpha_{t+1} S_t + \beta_{t+1} B_t. \quad (2.33)$$

The last equation does not involve the contingent claim since this is not traded after time zero. The value of the portfolio at time T is

$$V_T(\psi) = \alpha_T S_T + \beta_T B_T + \gamma_1 X. \quad (2.34)$$

An *arbitrage opportunity* in the stock-bond-contingent claim market is a trading strategy ψ such that $V_0(\psi) = 0$, $V_T(\psi) \geq 0$ and $E[V_T(\psi)] > 0$.

Theorem 2.2.4. *Let $X^* = X/(1+r)^T$. Then $V_0 = E^{p^*}[X^*]$ is the unique arbitrage free initial price for the European contingent claim X .*

Proof. The proof is very similar to that of Theorem 2.2.1.

Let $\phi^* = \{(\alpha_t^*, \beta_t^*), t = 1, \dots, T\}$ denote the replicating strategy for the contingent claim X obtained by applying the inductive procedure described above (cf. (2.26)–(2.27)). Then the value process $V = \{V_t(\phi^*), t = 0, 1, \dots, T\}$ for ϕ^* satisfies (2.28) for $t = 0, 1, 2, \dots, T$. Note that, by (2.29), V_0 is the initial value of this replicating strategy.

We use the existence of the replicating strategy to show that if the initial price C_0 of the contingent claim is not V_0 , then there is an arbitrage opportunity in the stock-bond-contingent claim market. Suppose $C_0 > V_0$. Then an investor could sell one contingent claim initially, use V_0 of the proceeds to invest in the stock-bond replicating strategy ϕ^* , and buy $C_0 - V_0$ additional units of bond at time zero and hold them over the entire period $(0, T]$. Thus, the trading strategy is $\psi = \{(\alpha_t^*, \beta_t^* + C_0 - V_0), t = 1, 2, \dots, T; \gamma_1 = -1\}$. This has initial value $V_0(\psi) = \alpha_1^* S_0 + \beta_1^* + C_0 - V_0 - C_0 = 0$, since $B_0 = 1$ and $V_0 = V_0(\phi^*) = \alpha_1^* S_0 + \beta_1^*$. The value of this portfolio at time T is

$$\begin{aligned} V_T(\psi) &= \alpha_T^* S_T + \beta_T^* B_T + (C_0 - V_0) B_T - X \\ &= X + (C_0 - V_0) B_T - X = (C_0 - V_0) B_T > 0, \end{aligned} \quad (2.35)$$

where we have used the fact that $\{(\alpha_t^*, \beta_t^*), t = 1, 2, \dots, T\}$ is a replicating strategy for the contingent claim X and so has value X at time T . Thus, ψ is an arbitrage opportunity. Similarly, if $C_0 < V_0$, the strategy $-\psi$ is an arbitrage opportunity.

Now suppose $C_0 = V_0$. We show that there is no arbitrage opportunity in stock, bond and contingent claim trading. Let $\psi = \{(\alpha_t, \beta_t), t = 1, 2, \dots, T; \gamma_1\}$ be a trading strategy in stock, bond and contingent claim with an initial value of zero and final value $V_T(\psi)$ that is a non-negative random variable. Then,

$$0 = V_0(\psi) = \alpha_1 S_0 + \beta_1 B_0 + \gamma_1 C_0, \quad (2.36)$$

$$0 \leq V_T(\psi) = \alpha_T S_T + \beta_T B_T + \gamma_1 X. \quad (2.37)$$

Note that $\phi = \{(\alpha_t, \beta_t), t = 1, 2, \dots, T\}$ is a trading strategy in stock and bond. Using the martingale property of $\{V_t^*(\phi), \mathcal{F}_t, t = 0, 1, \dots, T\}$ under p^* , on setting $V_t^*(\psi) = V_t(\psi)/(1+r)^t$ for $t = 0, 1, \dots, T$, we have

$$\begin{aligned} \frac{1}{(1+r)^T} E^{p^*}[V_T(\psi)] &= E^{p^*}[V_T^*(\psi)] \\ &= E^{p^*}[V_T^*(\phi)] + E^{p^*}[\gamma_1 X^*] \\ &= E^{p^*}[V_0^*(\phi)] + \gamma_1 E^{p^*}[X^*] \\ &= \alpha_1 S_0^* + \beta_1 B_0^* + \gamma_1 C_0 \\ &= V_0(\psi) = 0 \end{aligned}$$

where we have used the facts that $\gamma_1 \in \mathcal{F}_0$, $C_0 = E^{p^*}[X^*]$, $S_0^* = S_0$, $B_0^* = B_0$, and $V_0(\psi) = 0$. Since it was assumed that $V_T(\psi) \geq 0$ and the probability associated with p^* gives positive probability to all possible values of $V_T(\psi)$, it follows that $V_T(\psi) = 0$ and hence $E[V_T(\psi)] = 0$. Thus, there cannot be any arbitrage opportunity when $C_0 = V_0$. $\square$

2.3. Pricing an American Contingent Claim

An *American contingent claim* (ACC) is represented by a (finite) sequence $Y = \{Y_t, t = 0, 1, \dots, T\}$ of real-valued random variables such that Y_t is $\mathcal{F}_t$ -measurable for $t = 0, 1, 2, \dots, T$. The random variable Y_t , $t = 0, 1, 2, \dots, T$, is interpreted as the payoff for the claim if the owner cashes it in at time t . The time at which the owner cashes in the claim is required to be a stopping time taking values in $\{0, 1, \dots, T\}$; i.e., a random variable $\tau : \Omega \rightarrow \{0, 1, \dots, T\}$ such that $\{\tau = t\} = \{\omega \in \Omega : \tau(\omega) = t\} \in \mathcal{F}_t$, $t = 0, 1, 2, \dots, T$. For $s, t \in \{0, 1, \dots, T\}$ such that $s \leq t$, let $\mathcal{T}_{[s,t]}$ denote the set of integer-valued stopping times that take values in the interval $[s, t]$. An example of an American contingent claim is an *American call option* with strike price K which has payoff $Y_t = (S_t - K)^+$ at time t , $t = 0, 1, 2, \dots, T$. Note that if $S_t \leq K$, cashing in the contingent claim at time t has an equivalent payoff to that obtained by not exercising the option at all. We have adopted this convention so that we can use one framework for treating all contingent claims, including options and contracts.

An important feature of an American contingent claim is that the buyer and the seller of such a derivative have different actions available to them — the buyer may cash in the claim at any stopping time $\tau \in \mathcal{T}_{[0,T]}$, whereas the seller seeks protection from the risk associated with all possible choices of the stopping time τ by the buyer. As with the pricing of European contingent claims, for the pricing of American contingent claims, an essential role will be played by a trading strategy that hedges the risk for the seller of an American contingent claim. However, unlike the European contingent claim setting, the seller will not always be able to exactly replicate the payoff of the American contingent claim at all times t . Instead, the seller of an American contingent claim seeks a *superhedging strategy* which is a (self-financing) trading strategy ϕ whose value is at least as great as the payoff of the American contingent claim at each time t .

More precisely, let $Y = \{Y_t, t = 0, 1, \dots, T\}$ be the payoff sequence for an American contingent claim (abbreviated as ACC). For $t = 0, 1, \dots, T$, let U_t denote the minimum amount of wealth that the seller of the ACC must have at time t in order to ensure that the seller has enough to cover the payoff if the buyer cashes in the claim at some stopping time $\tau \in \mathcal{T}_{[t,T]}$. A superhedging strategy for the seller is a (self-financing) trading strategy

$\phi = \{(\alpha_t, \beta_t), t = 1, 2, \dots, T\}$ with value $V_t(\phi)$ at time $t = 0, 1, \dots, T$, such that $U_t \leq V_t(\phi)$ for $t = 0, 1, 2, \dots, T$. Such a ϕ can be constructed stepwise by proceeding backwards through the binary tree. In order to see this, note that $U_T = Y_T$. Conditioned on $\mathcal{F}_{T-1}$ (i.e., given $S_0, S_1, \dots, S_{T-1}$), let U_T^u denote the amount that the seller of the ACC must cover at time T if the value of the stock at that time is $S_{T-1}u$ and let U_T^d denote the amount that the seller must cover at time T if the value of the stock at time T is $S_{T-1}d$. Thus, in the same manner as for the case of the European contingent claim, conditioned on $\mathcal{F}_{T-1}$, the minimum amount of wealth needed at time $T-1$ to cover the possible payoff of the ACC at time T is

$$\frac{1}{1+r} E^{p^*} [U_T \mid \mathcal{F}_{T-1}],$$

and the payoff at time T can be manufactured from this amount using $\mathcal{F}_{T-1}$ -measurable allocations $(\tilde{\alpha}_T, \tilde{\beta}_T)$ to stock and bond over the time interval $(T-1, T]$ determined by the relations:

$$\tilde{\alpha}_T S_T + \tilde{\beta}_T B_T = U_T, \quad (2.38)$$

$$\tilde{\alpha}_T S_{T-1} + \tilde{\beta}_T B_{T-1} = \frac{1}{1+r} E^{p^*} [U_T \mid \mathcal{F}_{T-1}]. \quad (2.39)$$

Now, at time $T-1$, the seller must have at least

$$U_{T-1} = \max \left\{ Y_{T-1}, \frac{1}{1+r} E^{p^*} [U_T \mid \mathcal{F}_{T-1}] \right\} \quad (2.40)$$

in order to cover the payoff Y_{T-1} associated with the buyer possibly cashing in the claim at time $T-1$, and to have sufficient wealth to produce a value at time T that is at least as large as the time T -payoff $U_T = Y_T$ of the ACC. Let

$$\tilde{\delta}_T = U_{T-1} - \frac{1}{1+r} E^{p^*} [U_T \mid \mathcal{F}_{T-1}]. \quad (2.41)$$

Then $\tilde{\delta}_T$ is the excess wealth in U_{T-1} over what is needed to cover the claim payoff at time T . Note that there is no excess if $Y_{T-1} < U_{T-1}$. Proceeding backwards inductively through the tree and repeating a very similar argument at each stage to that applied for the time interval $(T-1, T]$, we see that for $t = 0, 1, \dots, T-1$, conditioned on $\mathcal{F}_t$, the amount of wealth needed at time t to cover possible payoff of the claim in $[t, T]$ is

$$U_t = \max \left\{ Y_t, \frac{1}{1+r} E^{p^*} [U_{t+1} \mid \mathcal{F}_t] \right\}. \quad (2.42)$$

Moreover, conditioned on $\mathcal{F}_t$ (i.e., given $S_0, S_1, \dots, S_t$) and letting U_{t+1}^u and U_{t+1}^d denote the two possible values of U_{t+1} corresponding to whether $S_{t+1} = S_t u$ or $S_{t+1} = S_t d$, the allocations in stock and bond, $(\tilde{\alpha}_{t+1}, \tilde{\beta}_{t+1})$ over

$(t, t + 1]$, that have value at time t given by

$$\frac{1}{1+r} E^{p^*} [U_{t+1} \mid \mathcal{F}_t] \quad (2.43)$$

and value U_{t+1} at time $t + 1$ are given by

$$\tilde{\alpha}_{t+1} = \frac{U_{t+1}^u - U_{t+1}^d}{(u-d)S_t} \quad (2.44)$$

$$\tilde{\beta}_{t+1} = \frac{1}{(1+r)^{t+1}} \left(\frac{uU_{t+1}^d - dU_{t+1}^u}{u-d} \right). \quad (2.45)$$

Let

$$\tilde{\delta}_{t+1} = U_t - \frac{1}{1+r} E^{p^*} [U_{t+1} \mid \mathcal{F}_t]. \quad (2.46)$$

Then $\tilde{\delta}_{t+1}$ is the excess wealth in U_t over what is needed to cover possible payoff of the claim in $[t + 1, T]$. Note from the definition of U_t that $\tilde{\delta}_{t+1} = 0$ if $Y_t < U_t$.

Assuming that $U_t, t = 0, 1, \dots, T$ and $\tilde{\alpha}_t, \tilde{\beta}_t, \tilde{\delta}_t, t = 1, 2, \dots, T$ are defined as above, let

$$\alpha_t^* = \tilde{\alpha}_t, \quad (2.47)$$

$$\beta_t^* = \tilde{\beta}_t + \sum_{s=1}^t \frac{\tilde{\delta}_s}{B_{s-1}}, \quad (2.48)$$

for $t = 1, 2, \dots, T$. For this, note that $\tilde{\delta}_s$ is the excess available at time $s - 1$ over the amount $\frac{1}{1+r} E^{p^*} [U_s \mid \mathcal{F}_{s-1}]$ that is needed at time $s - 1$ to fund $(\tilde{\alpha}_s, \tilde{\beta}_s)$. We think of this excess as being invested in the bond for the time period $(s - 1, T]$. The cost of unit of bond is B_{s-1} at time $s - 1$, and so the number of units of the bond bought with $\tilde{\delta}_s$ will be $\tilde{\delta}_s / B_{s-1}$.

By construction, for $t = 1, \dots, T$,

$$\tilde{\alpha}_t S_t + \tilde{\beta}_t B_t = U_t.$$

The strategy $\phi^* = \{(\alpha_t^*, \beta_t^*), t = 1, \dots, T\}$ for trading in stock and bond is self-financing since for $t = 1, \dots, T-1$,

$$\begin{aligned}
\alpha_{t+1}^* S_t + \beta_{t+1}^* B_t &= \tilde{\alpha}_{t+1} S_t + \tilde{\beta}_{t+1} B_t + \left(\sum_{s=1}^{t+1} \frac{\tilde{\delta}_s}{B_{s-1}} \right) B_t \\
&= \frac{1}{1+r} E^{p^*} [U_{t+1} \mid \mathcal{F}_t] + \left(\sum_{s=1}^t \frac{\tilde{\delta}_s}{B_{s-1}} \right) B_t + \tilde{\delta}_{t+1} \\
&= U_t + \left(\sum_{s=1}^t \frac{\tilde{\delta}_s}{B_{s-1}} \right) B_t \\
&= \tilde{\alpha}_t S_t + \tilde{\beta}_t B_t + \left(\sum_{s=1}^t \frac{\tilde{\delta}_s}{B_{s-1}} \right) B_t \\
&= \alpha_t^* S_t + \beta_t^* B_t.
\end{aligned}$$

For the first equality above we used the definition of $(\alpha_{t+1}^*, \beta_{t+1}^*)$ (cf. (2.47)–(2.48)). For the second equality we used the fact that $(\tilde{\alpha}_{t+1}, \tilde{\beta}_{t+1})$ is designed to have the value given by (2.43) at time t , and we split off the last term from the sum in the previous line. For the third equality we used the definition (2.46) of $\tilde{\delta}_{t+1}$. For the fourth equality we used the fact that (α_t, β_t) is designed to have the value U_t at time t . The fifth equality simply uses the definition (2.47)–(2.48) of (α_t^*, β_t^*) .

The initial value of the trading strategy ϕ^* is

$$\begin{aligned}
V_0(\phi^*) &= \alpha_1^* S_0 + \beta_1^* B_0 \\
&= \tilde{\alpha}_1 S_0 + \tilde{\beta}_1 B_0 + \tilde{\delta}_1 \\
&= \frac{1}{1+r} E^{p^*} [U_1] + U_0 - \frac{1}{1+r} E^{p^*} [U_1] \\
&= U_0.
\end{aligned}$$

Here we have used the fact that $\mathcal{F}_0$ is trivial, and so conditional expectations with respect to this σ -algebra simplify to ordinary expectations. The value of ϕ^* at time $t = 1, 2, \dots, T$ is

$$V_t(\phi^*) = \tilde{\alpha}_t S_t + \tilde{\beta}_t B_t + \left(\sum_{s=1}^t \frac{\tilde{\delta}_s}{B_{s-1}} \right) B_t \geq U_t.$$

Consider the stopping time

$$\tau^* = \min\{s \geq 0 : U_s = Y_s\}. \quad (2.49)$$

Note that since $Y_s \leq U_s$ for $s = 0, 1, \dots, T$, $Y_s < U_s$ for $0 \leq s < \tau^*$, and so $\tilde{\delta}_s = 0$ for $1 \leq s \leq \tau^*$, and hence

$$V_t(\phi^*) = U_t \quad \text{for } 0 \leq t \leq \tau^*. \quad (2.50)$$

For $t = 0, 1, \dots, T$, define the discounted random variables

$$Y_t^* = (1+r)^{-t}Y_t, \quad U_t^* = (1+r)^{-t}U_t. \quad (2.51)$$

Then for $t = 0, 1, \dots, T-1$,

$$U_t^* = \max \left\{ Y_t^*, E^{p^*} [U_{t+1}^* | \mathcal{F}_t] \right\}, \quad (2.52)$$

and $U_T^* = Y_T^*$. The process $\{U_t^*, t = 0, 1, \dots, T\}$ is called the *Snell envelope* of $\{Y_t^*, t = 0, 1, \dots, T\}$, and it has the following properties.

Lemma 2.3.1.

- (i) Under p^* , $U^* = \{U_t^*, \mathcal{F}_t, t = 0, 1, \dots, T\}$ is the smallest supermartingale such that $U_t^* \geq Y_t^*$ for $t = 0, 1, \dots, T$.
- (ii) For $t = 0, 1, \dots, T$,

$$U_t^* = \max_{\tau \in \mathcal{T}_{[t, T]}} E^{p^*} [Y_\tau^* | \mathcal{F}_t]. \quad (2.53)$$

- (iii) For $t = 0, 1, \dots, T$,

$$\tau^*(t) \equiv \min\{v \geq t : U_v^* = Y_v^*\}$$

is an element of $\mathcal{T}_{[t, T]}$ that achieves the maximum in the right side of (2.53).

Proof. Throughout this proof, all expectations and conditional expectations are to be computed under p^* . For (i), the supermartingale property and the inequality are immediate consequences of (2.52). To show that U^* is the smallest supermartingale satisfying the inequality, suppose that

$$W = \{W_t, \mathcal{F}_t, t = 0, 1, \dots, T\}$$

is another supermartingale such that $W_t \geq Y_t^*$, $t = 0, 1, \dots, T$. Then $W_T \geq Y_T^* = U_T^*$. For a proof by backwards induction, suppose that $s \in \{0, 1, \dots, T-1\}$ and $W_t \geq U_t^*$ for $t = s+1, \dots, T$. Then by the supermartingale property of W and the induction assumption,

$$W_s \geq E^{p^*} [W_{s+1} | \mathcal{F}_s] \geq E^{p^*} [U_{s+1}^* | \mathcal{F}_s].$$

Since by assumption, $W_s \geq Y_s^*$, we have that W_s is greater than or equal to the maximum of Y_s^* and $E^{p^*} [U_{s+1}^* | \mathcal{F}_s]$, which equals U_s^* . This completes the induction step and so $W_t \geq U_t^*$ for $t = T, T-1, \dots, 1, 0$.

Before proceeding with the main proof of parts (ii) and (iii), we first verify that $\tau^*(t)$ is a member of $\mathcal{T}_{[t, T]}$ for $t = 0, 1, \dots, T$. Fix $t \in \{0, 1, \dots, T\}$. Then $\tau^*(t) \geq t$ by the form of the definition and $\tau^*(t) \leq T$ since $U_T^* = Y_T^*$. Now Y_v^* and U_v^* are $\mathcal{F}_v$ -measurable for each $v \in \{0, 1, \dots, T\}$, and so it follows that for each $s \in \{t, t+1, \dots, T\}$,

$$\{\tau^*(t) = s\} = \{Y_v^* < U_v^* \text{ for } v \in [t, s) \text{ and } Y_s^* = U_s^*\} \in \mathcal{F}_s.$$

Hence $\tau^*(t)$ is a stopping time.

We prove the remainder of (ii) and (iii) together, using backwards induction again. For $t = T$, both (ii) and (iii) are easy to show since $\mathcal{T}_{[T,T]} = \{T\}$, Y_T^* is $\mathcal{F}_T$ -measurable, and $U_T^* = Y_T^*$. For the induction step, assume that for some $s \in \{0, 1, \dots, T-1\}$ both (ii) and (iii) hold for $t = s+1, s+2, \dots, T$. By (2.52) with $t = s$, and (ii) with $t = s+1$, for each $\sigma \in \mathcal{T}_{[s+1,T]}$ we have

$$\begin{aligned} U_s^* &\geq \max \left\{ Y_s^*, E^{p^*} \left[E^{p^*} [Y_\sigma^* | \mathcal{F}_{s+1}] | \mathcal{F}_s \right] \right\} \\ &= \max \left\{ Y_s^*, E^{p^*} [Y_\sigma^* | \mathcal{F}_s] \right\}. \end{aligned} \quad (2.54)$$

Here we have used the tower property of conditional expectations. For $\tau \in \mathcal{T}_{[s,T]}$,

$$Y_\tau^* = 1_{\{\tau=s\}} Y_s^* + 1_{\{\tau \geq s+1\}} Y_{\tau \vee (s+1)}^*,$$

where $\tau \vee (s+1) = \max\{\tau, s+1\}$. Since τ is a stopping time taking integer values in $[s, T]$, $1_{\{\tau \geq s+1\}} = 1_{\{\tau=s\}^c} \in \mathcal{F}_s$, where the superscript c denotes complement. Then using basic properties of conditional expectations we have

$$\begin{aligned} E^{p^*} [Y_\tau^* | \mathcal{F}_s] &= 1_{\{\tau=s\}} Y_s^* + 1_{\{\tau \geq s+1\}} E^{p^*} [Y_{\tau \vee (s+1)}^* | \mathcal{F}_s] \\ &\leq \max \left\{ Y_s^*, E^{p^*} [Y_{\tau \vee (s+1)}^* | \mathcal{F}_s] \right\}. \end{aligned} \quad (2.55)$$

Since $\tau \vee (s+1) \in \mathcal{T}_{[s+1,T]}$, it follows from (2.54) and (2.55) that

$$U_s^* \geq E^{p^*} [Y_\tau^* | \mathcal{F}_s],$$

and since $\tau \in \mathcal{T}_{[s,T]}$ was arbitrary, we conclude that

$$U_s^* \geq \max_{\tau \in \mathcal{T}_{[s,T]}} E^{p^*} [Y_\tau^* | \mathcal{F}_s]. \quad (2.56)$$

The proof of (ii) and (iii) for $t = s$ will be complete once we verify that the maximum in the right member of (2.56) is achieved at $\tau = \tau^*(s)$ and the value of this maximum is U_s^* . By (2.52) with $t = s$ and (iii) with $t = s+1$, we have

$$\begin{aligned} U_s^* &= \max \left\{ Y_s^*, E^{p^*} \left[E^{p^*} [Y_{\tau^*(s+1)}^* | \mathcal{F}_{s+1}] | \mathcal{F}_s \right] \right\} \\ &= \max \left\{ Y_s^*, E^{p^*} [Y_{\tau^*(s+1)}^* | \mathcal{F}_s] \right\}. \end{aligned} \quad (2.57)$$

On $\{\tau^*(s) = s\}$, $Y_s^* = U_s^*$, and on $\{\tau^*(s) \geq s+1\}$, $Y_s^* < U_s^*$. Thus,

$$U_s^* = 1_{\{\tau^*(s)=s\}} Y_s^* + 1_{\{\tau^*(s) \geq s+1\}} E^{p^*} [Y_{\tau^*(s+1)}^* | \mathcal{F}_s].$$

Moreover, on $\{\tau^*(s) \geq s+1\}$, $\tau^*(s) = \tau^*(s+1)$ and so the above simplifies to

$$U_s^* = E^{p^*} [Y_{\tau^*(s)}^* | \mathcal{F}_s].$$

Here we have used the fact that $\tau^*(s)$ is a stopping time with integer values in $[s, T]$ and so $\{\tau^*(s) \geq s+1\} = \{\tau^*(s) = s\}^c \in \mathcal{F}_s$. This completes the proof of (ii) and (iii) for $t = s$, and so the desired result follows by induction. $\square$

Recall the definition of τ^* from (2.49) and note that

$$\tau^* = \tau^*(0).$$

Then we have the following.

Lemma 2.3.2. *Under p^* , $\{U_{t \wedge \tau^*}^*, \mathcal{F}_t, t = 0, 1, \dots, T\}$ is a martingale.*

Proof. Since τ^* is a stopping time, $U_{t \wedge \tau^*}^* \in \mathcal{F}_t$ for $t = 0, 1, \dots, T$ and the probability space is finite, the integrability of $U_{t \wedge \tau^*}^*$ is automatic for each t . Fix $t \in \{0, 1, \dots, T-1\}$. On $\{\tau^* > t\}$, $Y_t^* < U_t^*$ and thus $U_t^* = E^{p^*}[U_{t+1}^* | \mathcal{F}_t]$ there. Thus,

$$\begin{aligned} U_{(t+1) \wedge \tau^*}^* - U_{t \wedge \tau^*}^* &= 1_{\{\tau^* > t\}} (U_{t+1}^* - U_t^*) \\ &= 1_{\{\tau^* > t\}} \left(U_{t+1}^* - E^{p^*}[U_{t+1}^* | \mathcal{F}_t] \right), \end{aligned} \quad (2.58)$$

and then taking conditional expectations with respect to $\mathcal{F}_t$ under p^* , using the fact that $\{\tau^* > t\} \in \mathcal{F}_t$, yields

$$E^{p^*}[U_{(t+1) \wedge \tau^*}^* - U_{t \wedge \tau^*}^* | \mathcal{F}_t] = 0.$$

$\square$

We now argue that U_0 is the unique arbitrage free initial price for the American contingent claim. For this, we need the notion of an arbitrage in a market where the stock and bond can be traded and the American contingent claim (ACC) can be bought or sold at time zero. For such a market, let the initial price of the ACC be a constant C_0 . There are two types of arbitrage opportunities: one for a seller and another for a buyer of the ACC. The *seller* of the ACC has an *arbitrage opportunity* if there is a trading strategy ϕ^s in stock and bond such that $V_0(\phi^s) = C_0$ and for all stopping times $\tau \in \mathcal{T}_{[0, T]}$,

$$V_\tau(\phi^s) - Y_\tau \geq 0 \quad \text{and} \quad E[V_\tau(\phi^s) - Y_\tau] > 0. \quad (2.59)$$

The *buyer* of the ACC has an *arbitrage opportunity* if there is a trading strategy ϕ^b in stock and bond such that $V_0(\phi^b) = -C_0$ and there exists a stopping time $\tau \in \mathcal{T}_{[0, T]}$ such that

$$V_\tau(\phi^b) + Y_\tau \geq 0 \quad \text{and} \quad E[V_\tau(\phi^b) + Y_\tau] > 0. \quad (2.60)$$

The price C_0 is arbitrage free if there is no arbitrage opportunity for a seller or buyer of the contingent claim at this price.

To take advantage of a seller's arbitrage opportunity, an investor could sell one ACC at time zero for C_0 and invest the proceeds C_0 according to the trading strategy ϕ^s until the claim is cashed in by the buyer at some stopping time τ . At time τ , the seller would give the buyer Y_τ to pay off the claim and have a resultant wealth at time τ of $V_\tau(\phi^s) - Y_\tau$. The seller could put this amount in bond for the time period $(\tau, T]$. This would result in a final value of $(V_\tau(\phi^s) - Y_\tau)(1 + r)^{T-\tau}$ which is non-negative and is strictly positive with positive probability (under p or p^*).

To take advantage of a buyer's arbitrage opportunity, an investor could buy one ACC at time zero for C_0 , and invest $-C_0$ according to ϕ^b until the time τ when the buyer cashes in the claim. The buyer would then have $V_\tau(\phi^b) + Y_\tau$ at time τ and could put this in bond for the time period $(\tau, T]$ so that the buyer's final wealth would be $(V_\tau(\phi^b) + Y_\tau)(1 + r)^{T-\tau}$ which is non-negative and is strictly positive with positive probability.

Remark. Here, in interpreting the definition of an arbitrage opportunity for the seller or buyer of an American contingent claim, we have imagined that the seller or buyer would continue investing in the primary market once the ACC had been cashed in. An advantage of this interpretation is that it facilitates comparison of payoffs at a fixed time T . Alternatively, one could imagine that once the ACC is cashed in, the buyer and seller do not further invest in the primary market. This is equivalent to them investing in a bond with zero interest rate once the ACC has been cashed in. With either interpretation, the mathematical definition of an arbitrage opportunity is the same.

The following lemma will be used in showing that an initial price of U_0 for the ACC is arbitrage free.

Lemma 2.3.3. *Let ϕ be a trading strategy in stock and bond with discounted value process*

$$\{V_t^*(\phi) = V_t(\phi)/(1 + r)^t, t = 0, 1, \dots, T\}.$$

Then, for any $\tau \in \mathcal{T}_{[0, T]}$,

$$E^{p^*}[V_\tau^*(\phi)] = V_0^*(\phi). \quad (2.61)$$

Proof. By Lemma 2.2.3, $\{V_t^*(\phi), \mathcal{F}_t, t = 0, 1, \dots, T\}$ is a martingale under p^* . Equation (2.61) then follows from Doob's stopping theorem since τ is a bounded stopping time. $\square$

Theorem 2.3.4. *The unique arbitrage free initial price for the American contingent claim is U_0 .*

Proof. We first show that the arbitrage free price cannot be anything other than U_0 ; i.e., we establish uniqueness of an arbitrage free initial price.

Suppose that $C_0 > U_0$. Then there is an arbitrage opportunity for the seller of the American contingent claim. To see this, let ϕ^s denote the trading strategy in stock and bond corresponding to investing U_0 according to the superhedging strategy ϕ^* , and, in addition, putting $C_0 - U_0$ in the bond at time zero and holding those units of bond until time T . Note that the initial value $V_0(\phi^s) = C_0$ and the value $V_t(\phi^*)$ of ϕ^* at time t is at least U_t for $t = 0, 1, \dots, T$. Then for any stopping time $\tau \in \mathcal{T}_{[0,T]}$,

$$\begin{aligned} V_\tau(\phi^s) - Y_\tau &= V_\tau(\phi^*) + (C_0 - U_0)B_\tau - Y_\tau \\ &\geq U_\tau + (C_0 - U_0)B_\tau - Y_\tau \\ &\geq (C_0 - U_0)B_\tau, \end{aligned} \tag{2.62}$$

where $(C_0 - U_0)B_\tau > 0$. Thus, there is an arbitrage opportunity for a seller of the American contingent claim.

On the other hand, suppose that $C_0 < U_0$. Then there is an arbitrage opportunity for the buyer of the American contingent claim. To see this, let ϕ^b denote the trading strategy corresponding to investing $-U_0$ according to the negative $-\phi^*$ of the superhedging strategy ϕ^* and investing $U_0 - C_0$ in the bond for all time. Furthermore, consider the stopping time τ^* (viewed as the time at which the ACC should be cashed in). Note that $V_t(-\phi^*) = -V_t(\phi^*)$ for $t = 0, 1, \dots, T$, and $V_0(\phi^b) = -C_0$. Then, by (2.50) and the definition of τ^* , we have that $V_{\tau^*}(\phi^*) = U_{\tau^*} = Y_{\tau^*}$, and so

$$V_{\tau^*}(\phi^b) + Y_{\tau^*} = -V_{\tau^*}(\phi^*) + (U_0 - C_0)B_{\tau^*} + Y_{\tau^*} = (U_0 - C_0)B_{\tau^*}.$$

Since $(U_0 - C_0)B_{\tau^*} > 0$, there is an arbitrage opportunity for a buyer of the American contingent claim.

Finally, suppose that $C_0 = U_0$. We need to show that $C_0 = U_0$ is arbitrage free; i.e., that there exists an arbitrage free initial price. We begin by showing that there is no arbitrage opportunity for a seller of an American contingent claim with $C_0 = U_0$. For a proof by contradiction, suppose that there exists a trading strategy ϕ^s such that $V_0(\phi^s) = U_0$ and for each $\tau \in \mathcal{T}_{[0,T]}$, (2.59) holds. Note that $\tau^* \in \mathcal{T}_{[0,T]}$. Then it follows from (2.59) with $\tau = \tau^*$ that $V_{\tau^*}(\phi^s) - Y_{\tau^*} \geq 0$ and that this inequality holds strictly with positive probability under p and hence under p^* . These same properties continue to hold after multiplying the random variable by the random discount factor $(1+r)^{-\tau^*}$, and so it follows that

$$E^{p^*} [V_{\tau^*}^*(\phi^s) - Y_{\tau^*}^*] > 0. \tag{2.63}$$

On the other hand, by (2.61), $E^{p^*} [V_{\tau^*}^*(\phi^s)] = V_0^*(\phi^s) = U_0^*$, and by (ii) and (iii) in Lemma 2.3.1, $E^{p^*} [Y_{\tau^*}^*] = U_0^*$. Combining these two properties, we obtain $E^{p^*} [V_{\tau^*}^*(\phi^s) - Y_{\tau^*}^*] = 0$, which contradicts (2.63). Therefore, no such ϕ^s exists and consequently there is no arbitrage opportunity for a seller of the American contingent claim.

Next we show that there is no arbitrage opportunity for a buyer of the American contingent claim with $C_0 = U_0$. For this, suppose that there exists a trading strategy ϕ^b such that $V_0(\phi^b) = -U_0$ and a stopping time $\tau \in \mathcal{T}_{[0,T]}$ such that (2.60) holds. For a contradiction, we will show that $E[V_\tau(\phi^b) + Y_\tau] \leq 0$, or equivalently that $E^{p^*}[V_\tau(\phi^b) + Y_\tau] \leq 0$, which in turn is equivalent to showing that $E^{p^*}[V_\tau^*(\phi^b) + Y_\tau^*] \leq 0$ (since we have assumed that $V_\tau(\phi^b) + Y_\tau \geq 0$). By (2.61), $E^{p^*}[V_\tau^*(\phi^b)] = V_0^*(\phi^b) = -U_0^*$. Moreover, by (ii) in Lemma 2.3.1, $E^{p^*}[Y_\tau^*] \leq U_0^*$. Therefore, $E^{p^*}[V_\tau^*(\phi^b) + Y_\tau^*] \leq 0$, and the desired contradiction is obtained. Consequently there is no arbitrage opportunity for a buyer of the American contingent claim. $\square$

2.4. Exercises

1. Consider a single period CRR model with $S_0 = \$100$, $S_1 = \$200$ or $\$50$, $r = 0.25$.

- Find the arbitrage free initial price of a European call option for one share of stock where the strike price is $K = \$100$ and the exercise time $T = 1$.
- Find a hedging strategy that replicates the value of the option described in (a).
- Suppose the option in (a) is initially priced at \$1 above the arbitrage free price. Describe a strategy (for trading in stock, bond and the option) that is an arbitrage.
- What is the arbitrage free initial price for a put option with the same strike price and exercise time as the call option described in (a)?

2. Consider a CRR model with $T = 2$, $S_0 = \$100$, $S_1 = \$200$ or $S_1 = \$50$, and an associated European call option with strike price $K = \$80$ and exercise time $T = 2$. Assume that the risk free interest rate is $r = 0.1$.

- Draw the binary tree and compute the arbitrage free initial price of the European call option at time zero.
- Determine an explicit hedging strategy for this option.
- Suppose that the option in (a) is initially priced \$2 below the arbitrage free price. Describe a strategy (for trading in stock, bond and the option) that is an arbitrage.
- Try to automate the pricing of a European call option in a computer program where T, S_0, u, d and K are variables.

3. Consider the same CRR model as in Exercise 2 but with an associated European contingent claim whose value at its exercise time $T = 2$ is given by

$$X = \max\{S_0, S_1, S_2\}. \quad (2.64)$$

Answer questions (a)–(d) of Exercise 2 for this contingent claim.

4. Consider the multi-period CRR model described in the text and a European contingent claim X . Suppose that trading in the contingent claim is allowed at each of the times $t = 0, 1, \dots, T - 1$, where the price of the contingent claim at time t is given by an $\mathcal{F}_t$ -measurable random variable C_t . Let ϕ^* be a replicating strategy (in stock and bond) for the contingent claim X . Show that

$$V_t(\phi^*) = \frac{1}{(1+r)^{T-t}} E^{p^*}[X | \mathcal{F}_t], \quad t = 0, 1, \dots, T. \quad (2.65)$$

Formulate a notion of an arbitrage opportunity in the market consisting of stock, bond and contingent claim. Show that the process $C_t = V_t(\phi^*)$, $t = 0, 1, \dots, T - 1$ defines the unique arbitrage free price process for the contingent claim.

5. Consider a CRR model with terminal time T . A European call option with strike price K and expiration date T has value $C_T = (S_T - K)^+$ at time T . A European put option with the same strike price and expiration date has value $P_T = (K - S_T)^+$ at time T . Note that $C_T - P_T = S_T - K$. Let $\{C_t, t = 0, 1, \dots, T - 1\}$ and $\{P_t, t = 0, 1, \dots, T - 1\}$ denote the arbitrage free price processes for the call and put options, respectively (cf. Exercise 4). Show that the following call-put parity relation holds:

$$C_t - P_t = S_t - K/(1+r)^{T-t}, \quad t = 0, 1, \dots, T - 1. \quad (2.66)$$

Note, you can show this using either a “no arbitrage” argument or the formulas for C_t, P_t resulting from Exercise 3.

6. Consider a CRR model with $T = 2$, $S_0 = \$100$, $S_1 = \$200$ or $S_1 = \$50$ (the same model as in Exercise 2). Now consider an American put option with strike price $K = \$120$. Assume that the risk free interest rate is $r = 0.1$.

- (a) Use a binary tree to compute the arbitrage free initial price of the American put option.
- (b) Determine an explicit superhedging strategy ϕ^* for this option.
- (c) Suppose that you can buy the American put option at time zero for \$1 less than its arbitrage free price. Explicitly describe a strategy that yields an arbitrage opportunity for a buyer of the American put option.
- (d) Try to automate the arbitrage free pricing of an American put option in a computer program where T, S_0, u, d and K are variables.

7. Prove that the arbitrage free initial price for an American call option is the same as the arbitrage free initial price for a European call option. Furthermore, show that with this initial (arbitrage free) price, if the buyer of an American call option waits until the expiration date to exercise the option, this prevents the seller from making a risk free profit (that is, it prevents a seller's arbitrage).