

Math 194: Homework 5

1. Consider the three period binomial model from Homework 2. For $t = 0, 1, 2, 3$, let $\mathcal{F}_t = \sigma\{S_s : 0 \leq s \leq t\}$, the σ -algebra generated by the stock price process up to time t . For each t , $\mathcal{F}_t$ contains all of the events that you can distinguish by knowing the stock price process up to time t . The entire collection of σ -algebras $\{\mathcal{F}_t, t = 0, 1, 2\}$ is called a filtration. Each $\mathcal{F}_t$ is generated by a partition $\mathcal{P}_t$ (a collection of disjoint events whose union is all of Ω).

(a) Describe the partitions $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ that generate each of the sigma algebras $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$, respectively.

(b) Compute each of the following conditional expectations:

$$E[S_3|\mathcal{F}_3] = E[S_3|\mathcal{P}_3]$$

$$E[S_3|\mathcal{F}_2] = E[S_3|\mathcal{P}_2]$$

$$E[S_3|\mathcal{F}_1] = E[S_3|\mathcal{P}_1]$$

$$E[S_2|\mathcal{F}_1] = E[S_2|\mathcal{P}_1]$$

$$E[S_1|\mathcal{F}_0] = E[S_1|\mathcal{P}_0] = E[S_1].$$

(c) Using the risk neutral probability p^* in place of p and letting $S_t^* = S_t/B_t$ for each t , compute

$$E^{p^*}[S_3^*|\mathcal{F}_2] = E^{p^*}[S_3^*|\mathcal{P}_2]$$

$$E^{p^*}[S_2^*|\mathcal{F}_1] = E^{p^*}[S_2^*|\mathcal{P}_1]$$

$$E^{p^*}[S_1^*|\mathcal{F}_0] = E^{p^*}[S_1^*|\mathcal{P}_0].$$

What conclusion can you draw from this about the process $\{S_t^*, \mathcal{F}_t, t \geq 0\}$ under the binomial model with p^* in place of p ?

2. Consider a stock price process $S = (S_0, S_1, S_2)$ defined on $\Omega = \{\omega_1, \omega_2, \dots, \omega_5\}$ by

$$S(\omega_1) = (8, 16, 32)$$

$$S(\omega_2) = (8, 16, 28)$$

$$S(\omega_3) = (8, 16, 10)$$

$$S(\omega_4) = (8, 4, 16)$$

$$S(\omega_5) = (8, 4, 4)$$

Note that this is *not* a binomial model. However, you can draw a tree to describe the possible paths followed by the stock price process. You should do this and then answer the following questions.

(a) For $t = 0, 1, 2$, let $\mathcal{F}_t = \sigma\{S_s : 0 \leq s \leq t\}$, the σ -algebra generated by the stock price process up to time t . Each $\mathcal{F}_t$ is generated by a partition $\mathcal{P}_t$ of Ω . Determine what $\mathcal{P}_t$ is for $t = 0, 1, 2$. Write down the list of sets that constitute $\mathcal{F}_t$, for $t = 0, 1, 2$.

(b) Let P be a probability measure defined on $(\Omega, \mathcal{F}_2)$ where $P(\{\omega_1\}) = 0.5$, $P(\{\omega_2\}) = 0.2$, $P(\{\omega_3\}) = 0.1$, $P(\{\omega_4\}) = 0.15$, $P(\{\omega_5\}) = 0.05$. Let E denote expectation under P . Determine the conditional expectations:

$$E[S_2|\mathcal{F}_1] = E[S_2|\mathcal{P}_1],$$

$$E[S_2|\mathcal{F}_0] = E[S_2|\mathcal{P}_0].$$

3. Consider the CRR (binomial) model described in Exercise 2 of Chapter 2 of the web notes as the model for a stock and a bond. A forward contract is to be offered under which the holder of a long position in the forward contract will buy 100 shares of stock at time $T = 2$ for a fixed price F . (Here F is the total amount to be paid for the 100 shares). Remember that no money changes hands at time zero when a forward contract is written. What value should F take in order that there is no arbitrage opportunity for the investor who holds a long or a short position in the contract? Explain your reasoning fully (in particular, identify an associated European contingent claim and derive the value of F using arbitrage pricing for the contingent claim).