

Math 194 Homework 6 Due February 24, 2012, noon.

1. Consider a sequence of independent, identically distributed random variables $\{\zeta_t, t = 1, 2, \dots, T\}$, where T is a finite positive integer, and $\zeta_t = 1$ with probability $1/2$ and $\zeta_t = -1$ with probability $1/2$ for $t = 1, 2, \dots, T$. Without loss of generality, we may take the probability space to be $(\Omega, \mathcal{F}, P)$ where Ω is the set of all possible paths followed by $\{\zeta_t, t = 1, 2, \dots, T\}$, $\mathcal{F}$ is all subsets of Ω , and P is the probability associated with the paths being equally likely, i.e., each path has probability $1/2^T$. Let $M_0 = 0$ and $M_t = \sum_{s=1}^t \zeta_s$ for $t = 1, 2, \dots, T$. You can think of M_t as representing the total gain up to time t in playing a game with an equal chance of gaining or losing a dollar at each step, regardless of what has happened previously in the game (this is an example of a ‘fair’ game). [Of course, if M_t is negative, the player has lost money in aggregate by time t , i.e., the player has a negative total gain.] For $t = 0, 1, \dots, T$, let $\mathcal{F}_t = \sigma\{\zeta_1, \dots, \zeta_t\}$ denote the σ -algebra generated by the incremental gains $\zeta_1, \dots, \zeta_t$ up to time t . By convention, $\mathcal{F}_0 = \{\emptyset, \Omega\}$. For each fixed t , $\mathcal{F}_t$ is the same as the sigma-algebra generated by $M_0, M_1, \dots, M_t$, because one can generate the latter sequence knowing the incremental steps $\zeta_1, \dots, \zeta_t$ and conversely, given $M_0, M_1, \dots, M_t$, one can recover the incremental steps $\zeta_1, \dots, \zeta_t$. In other words, the sequence $\{\zeta_1, \dots, \zeta_t\}$ provides the same information as $\{M_0, M_1, \dots, M_t\}$.

Prove that $\{M_t, \mathcal{F}_t, t = 0, 1, 2, \dots, T\}$ is a martingale on $(\Omega, \mathcal{F}, P)$.

2. For this problem, you may assume that you are in the context of the binomial model, although your proof will not need this property. Suppose that $\{M_t, \mathcal{F}_t, t = 0, 1, 2, \dots, T\}$ is a (discrete time) martingale on a probability space $(\Omega, \mathcal{F}, P)$ where $\{\mathcal{F}_t, t = 0, 1, \dots, T\}$ is a given filtration. In particular, suppose that for each $t = 0, 1, 2, \dots, T-1$,

$$E[M_{t+1} | \mathcal{F}_t] = M_t.$$

Starting from this property, give a formal proof of the property described in class that for any $0 \leq s < t \leq T$,

$$E[M_t | \mathcal{F}_s] = M_s.$$

Hint: give a formal proof by mathematical induction and use the tower property of conditional expectations. You may find it helpful to fix s , let $n = t - s$, and give a proof by induction for $n = 1, 2, \dots, T - s$.

3. For the following exercise, you should assume that the stock price follows a (continuous) Black-Scholes model so that under the risk neutral probability P^* , for any $t \in [0, T]$,

$$S_t = S_0 \exp \left(\left(r - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right),$$

where W_t has a normal distribution with mean 0 and variance t . Then if X represents the value at time T for a European contingent claim and X^* (or equivalently X^*) has finite mean under P^* , then the arbitrage free initial price for the contingent claim is given by

$$E^{P^*}[X^*]$$

where $X^* = Xe^{-rT}$.

If X corresponds to a European call option with strike price K , i.e., $X = (S_T - K)^+$, then

$$E^{P^*}[X^*] = S_0\Phi(d_+) - e^{-rT}K\Phi(d_-)$$

where Φ is the cumulative standard normal distribution function and

$$d_+ = \frac{\log(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_- = \frac{\log(S_0/K) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

Suppose that time is measured in years and that the riskless interest rate is 2% per year (compounded continuously). Assume that the volatility parameter $\sigma = 0.1$ and the initial stock price is \$15.

- (a) Consider a European call option on the stock with a strike price of \$14 and an expiration date of three months from the initial time. Compute the initial arbitrage free price for this option.
- (b) Consider a European contingent claim that expires 3 months after the initial time and which at expiry is worth \$1 if the stock price at that time is more than \$12 and is worth zero otherwise. Write down a formula for the value of the contingent claim at the expiration time in terms of the stock price. Compute the initial arbitrage free price for this contingent claim.

4. This exercise will require you to obtain historical data on the daily closing prices of Apple stock. Such data can be readily obtained from the Yahoo finance web site (a link to this is available from the links to related web sites on the Math 194 web site).

Consider a binomial model with $p = 0.5$, i.e., the stock price process for $t = 0, 1, \dots, T$ is given by

$$S_t = S_{t-1}\xi_t, \quad t = 1, \dots, T,$$

where S_0 is a positive constant, $\{\xi_t, t = 1, \dots, T\}$ is a sequence of i.i.d. random variables with

$$P(\xi_t = u) = p, \quad P(\xi_t = d) = 1 - p,$$

$0 < d < 1 + r < u$, and r is the risk-free interest rate. Let

$$X_t = \frac{S_t - S_{t-1}}{S_{t-1}} = \xi_t - 1, \quad t = 1, \dots, T.$$

Then, with $p = 0.5$, the mean and variance of X_t are given by

$$(1) \quad \mu = E^p[X_t] = pu + (1 - p)d - 1 = \frac{u + d}{2} - 1,$$

$$(2) \quad Var = E^p[(X_t - \mu)^2] = \frac{(u - d)^2}{4},$$

respectively.

Suppose that you have a sample of closing stock prices $\tilde{S}_0, \tilde{S}_1, \dots, \tilde{S}_T$ for times $t = 0, 1, \dots, T$. For $t = 0, 1, \dots, T$, let

$$\tilde{X}_t = \frac{\tilde{S}_t - \tilde{S}_{t-1}}{\tilde{S}_{t-1}}.$$

Then, the usual (unbiased) statistical estimators for μ and Var are given by

$$\bar{\mu} = \frac{1}{T} \sum_{t=1}^T \tilde{X}_t, \quad \overline{Var} = \frac{1}{T-1} \sum_{t=1}^T (\tilde{X}_t - \bar{\mu})^2.$$

- (a) Compute these estimators using the daily closing stock price data for Apple (ticker symbol AAPL), commencing with November 14, 2011 and ending with February 10, 2012.

You can obtain this historical data in downloadable Excel spreadsheet form from the Yahoo finance website <http://finance.yahoo.com/>. Once there, type in the ticker symbol AAPL, select “Historical Prices” in the left side column. Then type in the range of dates for which you want data and select “Daily”, then click on “Get Prices”. The historical data for the range of dates will be displayed and can be saved in spreadsheet format as a .csv file by clicking on “Download Spreadsheet Format” at the bottom of the table. Once you have saved the data to a file, you can open that file using Excel (note that it has a .csv extension, not a .xls extension). You can then use the built in Excel functions Average

and Var to compute the above estimators for a column of data – you can also use $Stdev$ instead of Var to obtain the square root of Var . Here, the natural choice for one unit of time is one trading day (ignore non-trading days such as weekends and holidays in counting time).

Using these estimates in place of μ and Var in (1) and (2) respectively, determine estimates for u and d .

- (b) Now consider the binomial model with these estimated values of u and d .

Based on this binomial model, find the arbitrage free price on February 10, 2012 for a European call option based on Apple stock with a strike price of \$500 and an expiration month of March.

Notes:

- (N1) To estimate the daily risk-free interest rate r , use the 3-month (13-week) T-bill rate on February 10, 2012 (take this to be 0.08% per annum). As an approximation, assume there are $65 = 13 \times 5$ trading days in a 3-month period, set $(1 + r)^{65} = 1 + \frac{0.0008}{4}$ and solve for r .
- (N2) Recall that all market-traded options (such as we are considering) effectively expire on the third Friday of their month of expiry. For March 2012, the third Friday is March 16, 2012.
- (N3) To compute the arbitrage free prices, you will probably want to use an Excel spreadsheet program, such as you have been working with on previous homework assignments. You will need to draw a bigger binomial tree with more steps in it for this – in particular, you will need to start further to the right in order to have room for your tree to spread.)
- (c) Comment on your findings, especially in comparison to the option prices listed for Apple corresponding to the close of trading on February 10, 2012 (the ask price for a European call option on Apple stock with an option strike price of \$500 and expiration month of March was \$13.85 on February 10, 2012.)