Lecture 5 **Math 286**

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Today: selected portions of Ch. 2 and 3.

Stochastic Integrals (Cont.)

(I) First define
$$\int_0^\infty X\,dB \text{ for suitable } X.$$

(i) $\tilde{\mathcal{E}}$ is a collection of simple integrands:

$$X(s,\omega) = \sum_{j=0}^{n-1} e_j(\omega) 1_{(t_j,t_{j+1}]}(s), \quad e_j \in b\mathcal{F}_{t_j}, \tag{1}$$

$$\int_0^\infty X(s,\omega)dB_s(\omega) = \sum_{j=0}^{n-1} e_j(\omega)(B_{t_{j+1}} - B_{t_j})(\omega).$$

Key concept to define a stochastic integral: isometry

$$\begin{array}{ccc} X & \longrightarrow & \int_0^\infty X\,dB \\ \big\downarrow & & \big\downarrow \\ \mathbb{R}_+ \times \Omega & & \Omega \end{array}$$

<u>Isometry:</u>

$$\begin{aligned} E\left[\left(\int_0^\infty X\,dB\right)^2\right] &= \sum_{j=0}^{n-1} E[e_j^2(B_{t_{j+1}} - B_{t_j})^2] \\ &\quad + \sum_{j=0}^{n-1}\sum_{k\neq j} E[e_j e_k (B_{t_{j+1}} - B_{t_j})(B_{t_{k+1}} - B_{t_k})]. \end{aligned}$$

Now,

$$\begin{aligned}
E[e_j^2(B_{t_{j+1}} - B_{t_j})^2] &= E[e_j^2 E[(B_{t_{j+1}} - B_{t_j})^2 | \mathcal{F}_{t_j}]] \\
&= E[e_j^2](t_{j+1} - t_j) \\
&= E\left[e_j^2(\omega)\int_{\mathbb{R}_+} 1_{(t_j,t_{j+1}]}(s)ds\right] \\
&= \int_{\mathbb{R}_+\times\Omega} e_j^2(\omega)1_{(t_j,t_{j+1}]}(s)d(\lambda\times P)(s,\omega)
\end{aligned}$$

where $\lambda =$ Lebesgue measure on $\mathbb{R}_+$. The sum of the above expression is (see (1) above):

$$= \int_{\mathbb{R}_+\times\Omega} X^2(s,\omega)d(\lambda\times P)(s,\omega).$$

We claim that the off diagonal term is zero. If $j < k$:

$$\begin{aligned}
E[e_j e_k(B_{t_{j+1}} - B_{t_j})(B_{t_{k+1}} - B_{t_k})] &= E[e_j e_k(B_{t_{j+1}} - B_{t_j})E[B_{t_{k+1}} - B_{t_k}|\mathcal{F}_{t_k}]] \\
&= 0.
\end{aligned}$$

Similarly for $j > k$.

Now,

$$X \in L^2(\mathbb{R}_+\times\Omega, \mathcal{B}\times\mathcal{F}, \lambda\times P) \longrightarrow \int_0^\infty X\,dB \in L^2(\Omega,\mathcal{F},P)$$

L^2 isometry:

$$\begin{aligned}
||X||_{L^2(\mathbb{R}_+\times\Omega,\mathcal{B}\times\mathcal{F},\lambda\times P)} &= \left(\int_{\mathbb{R}_+\times\Omega} X^2 d(\lambda\times P)\right)^{1/2} \\
&= \left(E\left[\left(\int_0^\infty X\,dB\right)^2\right]\right)^{1/2} \\
&= \left\|\int_0^\infty X\,dB\right\|_{L^2(\Omega,\mathcal{F},P)} \quad \text{for } X \in \tilde{\mathcal{E}}.
\end{aligned}$$

Let $X \in \overline{(\tilde{\mathcal{E}})}$ where the overline denotes the closure in $L^2(\mathbb{R}_+\times\Omega, \mathcal{B}\times\mathcal{F}, \lambda\times P)$. Using the isometry, we can extend the integral $\int_0^\infty X\,dB$ from $X \in \tilde{\mathcal{E}}$ to any $X \in \overline{(\tilde{\mathcal{E}})} \subset L^2(\mathbb{R}_+\times\Omega, \mathcal{B}\times\mathcal{F}, \lambda\times P)$.

$\tilde{\mathcal{E}}$ = simple integrands defined on $\mathbb{R}_+ \times \Omega$.

$\mathcal{P}$ = predictable σ-algebra is the smallest σ-algebra on $\mathbb{R}_+ \times \Omega$ such that each $X \in \tilde{\mathcal{E}}$ is $\mathcal{P}$-measurable. Or the σ-algebra generated by left continuous adapted processes.

$\mathcal{N}$= (null sets) all $\mathcal{B}\times\mathcal{F}$-measurable sets in $\mathbb{R}_+\times\Omega$ that have $(\lambda\times P)$-measure zero.

$\tilde{\mathcal{P}}$ = smallest σ-algebra containing $\mathcal{P}$ and $\mathcal{N}= \mathcal{P} \vee \mathcal{N}$.

Lemma 1: Suppose $Z : \mathbb{R}_+ \times \Omega \longrightarrow \mathbb{R}$ is $\mathcal{B} \times \mathcal{F}$-measurable. Then Z is $\tilde{\mathcal{P}}$-measurable if and only if there is a $\mathcal{P}$-measurable $X : \mathbb{R}_+ \times \Omega \longrightarrow \mathbb{R}$ such that $(\lambda \times P)(X \neq Z) = 0$.

$L^2(\mathbb{R}_+ \times \Omega, \mathcal{B} \times \mathcal{F}, \lambda \times P)$: X and Z are indistinguishable in this L^2 space. If Z is in this L^2 space, so is X and vice-versa.

Lemma 2: Any $\mathcal{B} \times \mathcal{F}$-measurable adapted $Z : \mathbb{R}_+ \times \Omega \longrightarrow \mathbb{R}$ is $\tilde{\mathcal{P}}$-measurable.

Lemma 3: $\tilde{\mathcal{E}}$ is dense in $L^2(\mathbb{R}_+ \times \Omega, \mathcal{P}, \lambda \times P)$.

<u>Summary</u>:

If Z is $\mathcal{B} \times \mathcal{F}$-measurable adapted and

$$E\left[\int_0^\infty Z^2(s,\omega)ds\right] < \infty \tag{2}$$

then we can define $\int_0^\infty ZdB$. By Lemma 2, Z is $\tilde{\mathcal{P}}$-measurable. By Lemma 1, there exists an $X \in \mathcal{P}$ s.t. $(\lambda \times P)(X \neq Z) = 0$. Then using (2), $X \in L^2(\mathbb{R}_+\times\Omega, \mathcal{P}, \lambda\times P)$. Hence by Lemma 3, $X \in \overline{(\tilde{\mathcal{E}})}$. Define: $\int_0^\infty ZdB = \int_0^\infty XdB$.

<u>Remark</u>:

If X is continuous and adapted, and

$$X^n(s,\omega) = \sum_{j=0}^{n-1} X_{t_j}(\omega) 1_{(t_j,t_{j+1}]}(s)$$

$$0 = t_0 < t_1 < \ldots < t_n < \ldots,$$

we can use X^n as a predictable approximation to X (it also works if X is left continuous, it would not work if X is right continuous).

(II) Define

$$\left\{ \int_0^t X\, dB, t \geqslant 0 \right\} \text{ for suitable } X$$

$\Lambda^2 = \left\{ X : \mathbb{R}_+ \times \Omega \longrightarrow \mathbb{R}, X \text{ is } \tilde{\mathcal{P}} - \text{measurable}, 1_{[0,t]} X \in \mathcal{L}^2 \text{ for each } t \geqslant 0 \right\}$

where $\mathcal{L}^2 = L^2(\mathbb{R}_+ \times \Omega, \tilde{\mathcal{P}}, \lambda \times P)$,

$1_{[0,t]}X \in \mathcal{L}^2$ is equivalent to $E\left[\int_0^t X^2(s,\omega) ds\right] < \infty$, and

(3)
$$(1_{[0,t]}X)(s,\omega) = \begin{cases} X(s,\omega) & \text{for } s \leqslant t, \\ 0 & \text{for } s > t. \end{cases}$$

For $X \in \Lambda^2$,

$$Y_t = \int_0^\infty (1_{[0,t]}X)_s dB_s, \quad \text{for each } t \geqslant 0,$$

is well defined since the integrand satisfies the properties we need.

Example: If $X = B$ (Brownian Motion), then

$$Y_t = \int_0^\infty (1_{[0,t]}B)_s dB_s.$$

Theorem: $\{Y_t,\ t \geqslant 0\}$ is a zero mean, martingale that has a continuous version.

Proof:

(i) If $X \in \tilde{\mathcal{E}}$, it is straightforward to verify these properties. Since then,

$$\begin{aligned} X(s,\omega) &= \sum_{j=0}^{n-1} e_j(\omega) 1_{(t_j, t_{j+1}]}(s) \\ Y_t &= \int_0^\infty 1_{[0,t]} X \, dB \\ &= \sum_{j=0}^{n-1} \left(e_j(\omega) \left(B_{t_{j+1} \wedge t} - B_{t_j \wedge t} \right)(\omega) \right) \end{aligned}$$

(ii) Zero mean and martingale property. Fix $t \geqslant 0$, $X \in \Lambda^2$ then $1_{[0,t]} X \in \mathcal{L}^2$. There is a sequance $\{Z^n\} \subset \tilde{\mathcal{E}}$ such that:

$$Z^n \longrightarrow 1_{[0,t]} X \quad \text{in } \mathcal{L}^2.$$

In fact,

$$1_{[0,t]} Z^n \longrightarrow 1_{[0,t]} X \quad \text{in } \mathcal{L}^2.$$

Then by the isometry,

$$Y_t^n = \int_0^\infty 1_{[0,t]} Z^n \, dB \longrightarrow \int_0^\infty 1_{[0,t]} X \, dB \quad \text{in } L^2(\Omega, \mathcal{F}, P).$$

Note that this also holds with s in place of t for any $0 \leq s \leq t$. Now, $Y_t^n \in \mathcal{F}_t$ for each n, by (i). There is a subsequence $Y_t^{n_l} \longrightarrow Y_t$ a.s. as $l \to \infty$. It follows (since $\mathcal{F}_t$ contains all P-null sets) that $Y_t \in \mathcal{F}_t$.

Fix $0 \leqslant s < t < \infty$. Then we have the following where the limits are in L^2:

$$\begin{aligned} E[Y_t | \mathcal{F}_s] &= E[\lim_n Y_t^n | \mathcal{F}_s] \\ &= \lim_n E[Y_t^n | \mathcal{F}_s] \\ &= \lim_n Y_s^n = Y_s \quad \text{a.s.} \end{aligned}$$

Finally, $E[Y_t] = E[Y_0] = 0$.

(iii) Continuous version: Fix $t \geqslant 0$. As in part (ii) $1_{[0,t]}Z^n \longrightarrow 1_{[0,t]}X$ in $\mathcal{L}^2$

$$Y^n \equiv \left\{ Y^n_s = \int_0^\infty 1_{[0,s]} Z^n dB,\ 0 \leqslant s \leqslant t \right\}$$

is a continuous L^2-martingale for each n.

For $n \geqslant m > 0$, $Y^n - Y^m$ is a continuous martingale. Using Doob's maximal inequality:

$$P\left(\sup_{0\leqslant s\leqslant t} |Y^n_s - Y^m_s| \geqslant \frac{1}{2^l}\right) \leqslant 2^{2l} E[|Y^n_t - Y^m_t|^2] \longrightarrow 0,$$

as $n,m \longrightarrow \infty$ for a fixed l. Can choose a subsequence, $Y^{n_l}_t$ s.t.

$$\begin{aligned} E[|Y^{n_{l+1}}_t - Y^{n_l}_t|^2] &\leqslant \tfrac{1}{2^{3l}} \\ \Longrightarrow P\left(\sup_{0\leqslant s\leqslant t} |Y^{n_{l+1}}_s - Y^{n_l}_s| \geqslant \frac{1}{2^l}\right) &\leqslant \tfrac{1}{2^l} \quad \text{for all } l. \\ \Longrightarrow \sum_l P(A_l) &< \infty, \end{aligned}$$

where

$$A_l = \left\{ \sup_{0\leqslant s\leqslant t} |Y^{n_{l+1}}_s - Y^{n_l}_s| \geqslant \frac{1}{2^l} \right\}.$$

Then by the Borel-Cantelli lemma,

$$\begin{aligned} P(A_l \text{ occurs for infinitely many } l) &= 0 \\ \Longrightarrow P(Y^{n_l} \text{ converges uniformly on } [0,t]) &= 1 \end{aligned}$$

Let $\tilde{Y}_s(\omega) = \overline{\lim_{l\longrightarrow\infty}}\, Y^{n_l}_s$, $0 \leqslant s \leqslant t$. Then, $\tilde{Y}$ is a.s. continuous on $[0,t]$ being an a.s. uniform limit of continuous processes. For $s \in [0,t]$

$$\begin{aligned} \tilde{Y}_s &= \lim_{l\longrightarrow\infty} Y^{n_l}_s \quad \text{a.s.} \\ Y_s &= \lim_{l\longrightarrow\infty} Y^{n_l}_s \quad \text{in } L^2 \\ \Longrightarrow \quad \tilde{Y}_s &= Y_s \quad \text{a.s.} \end{aligned}$$

Hence, $\tilde{Y}$ is a continuous version of Y on $[0,t]$.