

**MATH 280C****HOMEWORK #1, DUE APRIL 6, 2011.**

Please hand in the starred problems.

In the following problems all random variables are defined on a single probability space  $(\Omega, \mathcal{F}, P)$  and  $\{\mathcal{F}_n, n = 0, 1, 2, \dots\}$  is a filtration of sub-sigma-algebras of  $\mathcal{F}$ .

1. Let  $\{X_n, \mathcal{F}_n, n = 0, 1, 2, \dots\}$  be a martingale. Let  $\xi_n = X_n - X_{n-1}$  for all  $n \geq 1$ . Suppose that  $E[X_0^2] < \infty$  and  $\sum_{n=1}^{\infty} E[\xi_n^2] < \infty$ . Prove that there is a random variable  $X_{\infty} \in \mathcal{F}_{\infty} \equiv \bigvee_{n=0}^{\infty} \mathcal{F}_n$  satisfying  $E[X_{\infty}^2] < \infty$  such that  $X_n \rightarrow X_{\infty}$  a.s. and in  $L^2$  as  $n \rightarrow \infty$ .

2\*. Suppose that  $\{Y_n, n = 0, 1, 2, \dots\}$  and  $Y$  are all random variables in  $L^1$ . Assume that  $Y_n \rightarrow Y$  in  $L^1$  as  $n \rightarrow \infty$  and let  $\mathcal{F}_{\infty} = \bigvee_{n=0}^{\infty} \mathcal{F}_n$ . Show that

$$E[Y_n | \mathcal{F}_n] \rightarrow E[Y | \mathcal{F}_{\infty}] \text{ in } L^1 \text{ as } n \rightarrow \infty.$$

3\*. Let  $\{\xi_n, n = 1, 2, \dots\}$  be a sequence of independent and identically distributed random variables with finite mean  $\mu$ . Let  $S_n = \sum_{i=1}^n \xi_i$  for  $n = 1, 2, \dots$ . Define  $X_{-n} = S_n/n$  and  $\mathcal{F}_{-n} = \sigma\{S_n, \xi_{n+1}, \xi_{n+2}, \dots\}$ , for  $n = 1, 2, \dots$ .

(a) Prove that  $\{X_{-n}, \mathcal{F}_{-n}, n = 1, 2, \dots\}$  is a backwards martingale. (Hint:  $E[\xi_j | \mathcal{F}_{-n}] = E[\xi_k | \mathcal{F}_{-n}]$  for  $j, k \leq n$ ,  $n = 1, 2, \dots$ , by symmetry.)

(b) Use the backwards martingale convergence theorem and Kolmogorov's zero-one law to prove that  $S_n/n \rightarrow \mu$  a.s. as  $n \rightarrow \infty$ .

4\*. Suppose that  $\{X_n, \mathcal{F}_n, n = 0, 1, 2, \dots\}$  is a submartingale or supermartingale. Prove that for any  $\lambda > 0$  and  $n \geq 0$ ,

$$\lambda P\left(\max_{0 \leq i \leq n} |X_i| \geq 3\lambda\right) \leq 4E(|X_0|) + 3E(|X_n|).$$

How does this simplify in the supermartingale case if  $X_n \geq 0$  for all  $n$ ? Hint: You may use Doob's decomposition theorem.