

Math 194: Homework 7, due Wednesday, March 7, 2007.

1. For this problem, you may assume that you are in the context of the binomial model, although your proof will not need this property. Suppose that $\{M_t, \mathcal{F}_t, t = 0, 1, 2, \dots, T\}$ is a (discrete time) martingale. In particular, suppose that for each $t = 1, 2, \dots, T$,

$$E[M_t | \mathcal{F}_{t-1}] = M_{t-1}.$$

Starting from this property, verify the property described in class that for any $0 \leq s < t \leq T$,

$$E[M_t | \mathcal{F}_s] = M_s.$$

Hint: use the tower property of conditional expectations.

2. For the following exercise, you should assume that the stock price follows a (continuous) Black-Scholes model so that under the risk neutral probability P^* , for any $t \in [0, T]$,

$$S_t = S_0 \exp \left(\left(r - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right),$$

where W_t has a normal distribution with mean 0 and variance t . Then if X represents the value at time T for a European contingent claim and X^* (or equivalently X^*) has finite mean under P^* , then the arbitrage free initial price for the contingent claim is given by

$$E^{P^*}[X^*]$$

where $X^* = X e^{-rT}$.

If X corresponds to a European call option with strike price K , i.e., $X = (S_T - K)^+$, then

$$E^{P^*}[X^*] = S_0 \Phi(d_+) - e^{-rT} K \Phi(d_-)$$

where Φ is the cumulative standard normal distribution function and

$$d_+ = \frac{\log(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$
$$d_- = \frac{\log(S_0/K) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

Suppose that time is measured in years and that the riskless interest rate is 5% per year (compounded continuously). Assume that the volatility parameter $\sigma = 0.20$ and the initial stock price is \$10.

- (a) Consider a European call option on the stock with an exercise price of \$12 and an expiration date of six months from the initial time. Compute the initial arbitrage free price for this option.

- (b) Consider a European contingent claim that expires six months after the initial time and which at expiry is worth \$1 if the stock price at that time is more than \$10 and is worth zero otherwise. Write down a formula for the value of the contingent claim at the expiration time in terms of the stock price. Compute the initial arbitrage free price for this contingent claim.