

Math 194: Homework 5, Problems 1, 2

1. Consider a stock price process $S = (S_0, S_1, S_2)$ defined on $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$ by

$$S(\omega_1) = (8, 16, 32)$$

$$S(\omega_2) = (8, 16, 28)$$

$$S(\omega_3) = (8, 15, 10)$$

$$S(\omega_4) = (8, 4, 16)$$

$$S(\omega_5) = (8, 4, 4)$$

$$S(\omega_6) = (8, 5, 4)$$

Note that this is not a binomial model.

- (a) Draw a tree to illustrate the paths followed by the stock price process.
- (b) For $t = 0, 1, 2$, $\mathcal{F}_t = \sigma\{S_s : 0 \leq s \leq t\}$, is the σ -algebra generated by the stock price process up to time t . The entire collection of σ -algebras $\{\mathcal{F}_t, t = 0, 1, 2\}$ is called a filtration. Each $\mathcal{F}_t$ is generated by a partition $\mathcal{P}_t$ of Ω . Determine $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2$. (Note, you do NOT need to write out what $\mathcal{F}_t$ is for each t .)
- (c) Let $\tau = \min\{t \geq 0 : S_t \geq 14\} \wedge 2$. The effect of the $\wedge$ (meaning min) operation here is that if there is no $t \in \{0, 1, 2\}$ at which $S_t \geq 14$, then $\tau = 2$. Write out what τ is as a function on Ω . Prove that τ is a stopping time.
- (d) Let $\eta = \max\{t \geq 0 : S_t > 4\} \wedge 2$. Write out what η is as a function on Ω . Prove that η is *not* a stopping time.

2. Consider a sequence of independent, identically distributed random variables $\{\zeta_t, t = 1, 2, \dots, T\}$, where T is a finite positive integer, and $\zeta_t = 1$ with probability $1/2$ and $\zeta_t = -1$ with probability $1/2$ for $t = 1, 2, \dots, T$. Without loss of generality, we may take the probability space to be $(\Omega, \mathcal{F}, P)$ where Ω is the set of all possible paths followed by $\{\zeta_t, t = 1, 2, \dots, T\}$, $\mathcal{F}$ is all subsets of Ω , and P is the probability associated with the paths being equally likely, i.e., each path has probability $1/2^T$. Let $M_0 = 0$ and $M_t = \sum_{s=1}^t \zeta_s$ for $t = 1, 2, \dots, T$. You can think of M_t as representing the total gain up to time t in playing a game with an equal chance of gaining or losing a dollar at each step, regardless of what has happened previously in the game (this is an example of a ‘fair’ game). [Of course, if M_t is negative, the player has lost money in aggregate by time t , i.e., the player has a negative total gain.] For $t = 0, 1, \dots, T$, let $\mathcal{F}_t = \sigma\{\zeta_1, \dots, \zeta_t\}$ denote the σ -algebra generated by the incremental gains $\zeta_1, \dots, \zeta_t$ up to time t . By convention, $\mathcal{F}_0 = \{\emptyset, \Omega\}$. For each fixed t , $\mathcal{F}_t$ is the same as the sigma-algebra generated by $M_0, M_1, \dots, M_t$, because one can generate the latter sequence knowing the incremental steps $\zeta_1, \dots, \zeta_t$ and conversely, given $M_0, M_1, \dots, M_t$, one can recover the incremental steps $\zeta_1, \dots, \zeta_t$.

Prove that $\{M_t, \mathcal{F}_t, t = 0, 1, 2, \dots, T\}$ is a martingale on $(\Omega, \mathcal{F}, P)$.