

Lecture 9

①

Recap:

Heat eq. with step function IC:

$$u_t - k u_{xx} = 0, \quad k > 0$$

$$x \in \mathbb{R}, \quad t > 0$$

$$u(x, 0) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

Solⁿ: Try $u(x, t) = f(z(x, t))$
for $z = \frac{x}{\sqrt{4kt}}$

$$\Rightarrow f'' + 2z f' = 0$$

Solⁿ: $F(x, t) = \frac{1}{2} \left(1 + \operatorname{erf} \left(\frac{x}{\sqrt{4kt}} \right) \right)$

where $\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2} ds$, $\operatorname{erf}(0) = 0$
 $\lim_{z \rightarrow \pm\infty} \operatorname{erf}(z) \rightarrow \pm 1$

Recall: If $v(x,t)$ solves heat eq. then: (2)

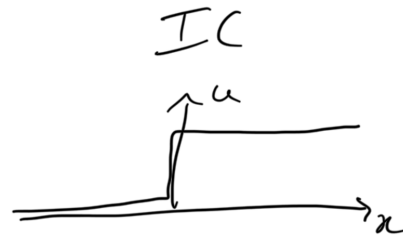
$v_n(x,t)$, $v(x-y,t)$, $\int v(x-y,t) \phi(y) dy$
are all solutions (for any function ϕ)

Goal: Solve $u_t - ku_{xx} = 0$, $x \in \mathbb{R}$, $t > 0$
 $u(x,0) = \phi(x)$ bounded

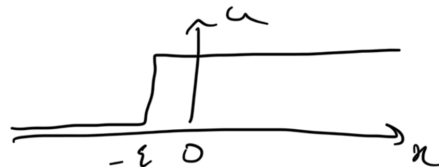
Strategy: Step-function IC $\rightarrow$ 'point source' IC
 $\rightarrow$ general IC $\phi(x)$

Solⁿ of $u_t - ku_{xx} = 0$

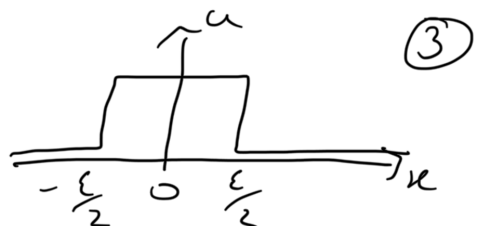
$F(x,t)$



$F(x+\epsilon, t)$



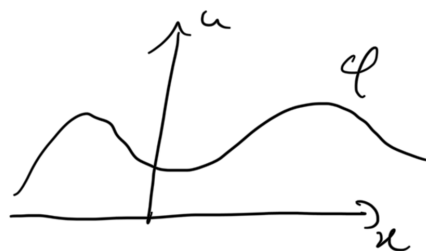
$$F(x + \frac{\epsilon}{2}, t) - F(x - \frac{\epsilon}{2}, t) \\ \approx \epsilon F_x(x, t)$$



[$F_x(x, t)$ solⁿ for point source at $x=0$]

IC ϕ can be thought of as a set of point sources of strength $\phi(y)$ at y for each $y \in \mathbb{R}$

$$\int_0^{\infty} \phi(y) F_x(x-y, t) dy$$



Heat eq. solⁿ

IC

[Amh: the point source function is also called the Dirac delta function, $\delta(x)$:

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases} \quad \text{s.t.} \quad \int_{-\infty}^{+\infty} \delta(x) dx = 1]$$

$$\text{Let } G(x, t) = F_x(x, t)$$

④

$$\Rightarrow G(x, t) = \partial_x \left(\frac{1}{2} \operatorname{erf} \left(\frac{x}{\sqrt{4kt}} \right) \right)$$

$$\Rightarrow G(x, t) = \frac{1}{\sqrt{4\pi kt}} \exp \left(\frac{-x^2}{4kt} \right)$$

G is called the heat kernel or fundamental solⁿ to the heat eq.

The solⁿ to the Cauchy problem:

$$u_t - ku_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0$$

$$u(x, t) = \phi(x), \quad \phi \text{ bounded}$$

is:

$$u(x, t) = \int_{-\infty}^{+\infty} G(x-y, t) \phi(y) dy$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi kt}} e^{\frac{-(x-y)^2}{4kt}} \phi(y) dy$$

(5)

The substitution $s = \frac{x-y}{\sqrt{4kt}}$, $ds = \frac{-1}{\sqrt{4kt}} dy$

gives the equivalent formula:

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-s^2} \varphi(x - s\sqrt{4kt}) ds$$

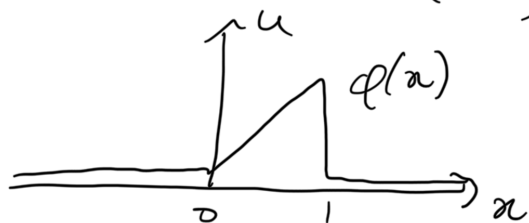
(Poisson integral representation)

Rmk.

- The solution is an integral representation
Often, solⁿ can't be found explicitly

Eg: Solve $u_t - ku_{xx} = 0$ subject to

$$u(x,0) = \varphi(x) = \begin{cases} x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$



Write solⁿ
in terms of erf

Solⁿ: $u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-s^2} \varphi(x - s\sqrt{4kt}) ds$ (6)

φ is non-zero between

$$x - s\sqrt{4kt} = 0 \Rightarrow s = \frac{x}{\sqrt{4kt}} = b$$

$$\text{and } x - s\sqrt{4kt} = l \Rightarrow s = \frac{x-l}{\sqrt{4kt}} = a$$

$$\Rightarrow u = \frac{1}{\sqrt{\pi}} \int_a^b e^{-s^2} (x - s\sqrt{4kt}) ds$$

$$= \frac{1}{\sqrt{\pi}} \left[x \int_a^b e^{-s^2} ds - \sqrt{4kt} \int_a^b s e^{-s^2} ds \right]$$

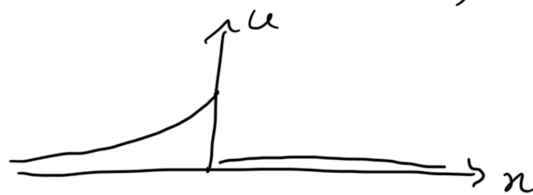
$$= \frac{1}{2} x (\operatorname{erf}(b) - \operatorname{erf}(a)) - \frac{\sqrt{4kt}}{\sqrt{\pi}} \left[-\frac{1}{2} e^{-s^2} \right]_a^b$$

$$= \frac{1}{2} x \left(\operatorname{erf}(b) - \operatorname{erf}(a) \right) + \sqrt{\frac{kt}{\pi}} \left(e^{-b^2} - e^{-a^2} \right) \quad (7)$$

$$\text{for } a = \frac{x-1}{\sqrt{4kt}}, \quad b = \frac{x}{\sqrt{4kt}}$$

Eg: Solve $u_t - k u_{xx} = 0$, $x \in \mathbb{R}$, $t > 0$

$$\text{with } u(x, 0) = \phi(x) = \begin{cases} e^{\lambda x}, & x \leq 0 \text{ (const. } \lambda > 0) \\ 0, & x > 0 \end{cases}$$



Write solⁿ
in terms of erf

$$\begin{aligned} \text{Sol}^n: u(x, t) &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-s^2} \phi(x - s\sqrt{4kt}) \, ds \\ &= \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{\infty} e^{-s^2} \cdot e^{\lambda x} \cdot e^{-\lambda s\sqrt{4kt}} \, ds \end{aligned}$$

$$= \frac{e^{\lambda x}}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{\infty} e^{-\frac{(s^2 + 2\lambda s \sqrt{kt})}{2}} ds \quad (8)$$

Complete the square: $s^2 + 2\lambda s \sqrt{kt}$
 $= (s + \lambda \sqrt{kt})^2 - \lambda^2 kt$

$$\Rightarrow u(x,t) = \frac{e^{\lambda x}}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{\infty} e^{-\frac{(s + \lambda \sqrt{kt})^2}{2}} e^{\frac{\lambda^2 kt}{2}} ds$$

Substitute $\theta = s + \lambda \sqrt{kt}$, $d\theta = ds$

$$\Rightarrow u(x,t) = \frac{e^{\lambda x + \lambda^2 kt}}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}} + \lambda \sqrt{kt}}^{\infty} e^{-\theta^2} d\theta$$

$$= \frac{1}{2} e^{\lambda x + \lambda^2 kt} \left(1 - \operatorname{erf} \left(\frac{x}{\sqrt{4kt}} + \lambda \sqrt{kt} \right) \right)$$

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Rmks:

- These solⁿs conserve mass: Suppose $u(x, t)$ is integrable and let

$$M(t) = \int_{-\infty}^{\infty} u(x, t) dx$$

$$\text{Then } \frac{dM}{dt} = \int_{-\infty}^{\infty} u_t dx = h \int_{-\infty}^{\infty} u_{xx} dx = h [u_x]_{-\infty}^{\infty}$$

$$\Rightarrow \frac{dM}{dt} = 0 \text{ if } u_x \rightarrow 0 \text{ as } x \rightarrow \pm \infty$$

$\Rightarrow u$ spreads out while conserving mass

- Solⁿs have infinite propagation speed.
Even if $u(x, 0) = 0$, we can have $u(x, t) \neq 0$ for large x