

Lecture 8

①

Recap:

- Cauchy problem for heat eq. (Logan 2.1)

Goal: solve for $u(x, t)$:

$$u_t = k u_{xx}, \quad k > 0$$

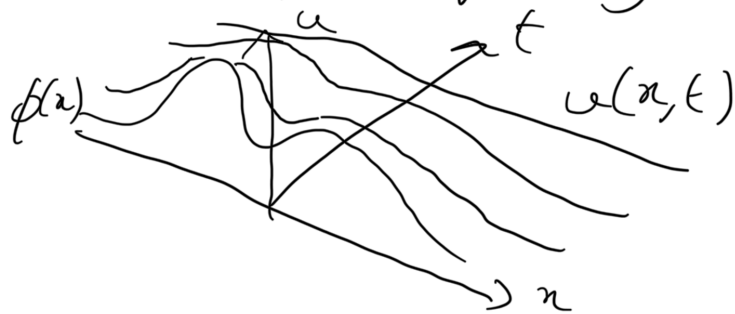
$$\text{domain: } x \in \mathbb{R}, \quad t > 0 \quad (*)$$

$$\text{IC: } u(x, 0) = \phi(x)$$

decay : u bounded as $x \rightarrow \pm \infty$
condition (equiv. $|x| \rightarrow \infty$)

u bounded $\Rightarrow \phi$ bounded

The solution will look $\phi(x)$ 'spreading out' over time:



- ②
- 1) First find the solution when $u(x, 0)$ is a step-function:
 - 2) Take linear combinations & translations of this solⁿ to solve the general Cauchy problem (*)

1) Step-function IC

$$u_t - k u_{xx} = 0, \quad k > 0, \quad x \in \mathbb{R}, \quad t > 0$$

$$u(x, 0) = \begin{cases} \alpha, & x > 0 \\ 0, & x < 0 \end{cases}$$

Dimensional analysis (magic trick)

The variables & constants in the problem are: x, t, u, k, α

Interpret u as temperature, x as length, t as time. Write $[\cdot]$ for dimensions:
 $[u] = \text{temp.}$ $[x] = \text{length},$ $[t] = \text{time}$

$$\Rightarrow [\alpha] = \text{temp.} \quad (\text{Since } u(x,0) = \alpha \text{ for } x > 0) \quad (3)$$

$$\& \quad [u_t] = \frac{\text{temp.}}{\text{time}}, \quad [u_{xx}] = \frac{\text{temp.}}{\text{length}^2}$$

$$\Rightarrow [k] = \left[\frac{u_t}{u_{xx}} \right] = \frac{\text{length}^2}{\text{time}}$$

$\Rightarrow$ There are 2 dimensionless quantities:

$\frac{1}{\alpha} u$ from temperature params

$\frac{x}{\sqrt{4kt}}$ from length, time params

[factor of 4 is for convenience]

Since $\frac{1}{\alpha} u$ is dimensionless it must be a function of dimensionless quantities

$$\Rightarrow \frac{1}{\alpha} u = f\left(\frac{x}{\sqrt{4kt}}\right)$$

for some f .

(4)

Remarks

- Any choice of dimensions for u, x, t would have led to this expression
- Having obtained this expression, we will forget about dimensions and set $\alpha = 1$

$$\Rightarrow u(x, t) = f\left(\frac{x}{\sqrt{4kt}}\right)$$

$$= f(z) \text{ for } z(x, t) = \frac{x}{\sqrt{4kt}}$$

- Trick: Substitute $u(x, t) = f(z)$ into heat eq. and hope we get an ODE for $f(z)$.

- We don't know that this will work yet! Above argument is slippery, full explanation involves 'symmetry' of PDE.

Heat eq. PDE $\rightarrow$ ODE

⑤

$$\text{Have } u_t - ku_{xx} = 0$$

$$u(x,0) = \begin{cases} 1, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

Look for solⁿs of form $u(x,t) = f(z(x,t))$

$$\text{for } z(x,t) = \frac{x}{\sqrt{4kt}}$$

Substitute into heat eq:

$$u_t = f'(z) \frac{\partial z}{\partial t} = -\frac{1}{2} \cdot \frac{x}{\sqrt{4kt^3}} f'(z)$$

$$= \frac{-z}{2t} f'(z)$$

$$u_x = f'(z) \frac{\partial z}{\partial x} = \frac{1}{\sqrt{4kt}} f'(z)$$

$$u_{xx} = \frac{1}{4kt} f''(z)$$

Then $u_t - ku_{xx} = 0$

(6)

$$\Rightarrow \frac{-z}{2t} f' - \frac{1}{4t} f'' = 0$$

$$\Rightarrow f'' + 2zf' = 0$$

- We've reduced the PDE for $u(x, t)$ to an ODE for $f(z)$.
- If we got an expression which involved other functions of x, t , we wouldn't have an ODE. E.g. looking for sol^{ns} of the form $u(x, t) = g(x^2 + t^2)$ wouldn't give an ODE for g .

Solⁿ of ODE:

$$f'' + 2zf' = 0$$

$$\text{let } g = f'$$

$$\Rightarrow \frac{dg}{dz} = -2zg$$

⑦

$$\Rightarrow \int \frac{1}{g} dg = \int -2z dz$$

$$\Rightarrow \log g = -z^2 + C$$

$$\Rightarrow g(z) = C_1 e^{-z^2}$$

$$\Rightarrow f'(z) = C_1 e^{-z^2} \quad \text{integrate}$$

$$\Rightarrow f'(z) = C_1 \int_0^z e^{-s^2} ds + C_2$$

$\downarrow$
 s is a dummy variable

Use $z = \frac{x}{\sqrt{4kt}}$

$$u(x,t) = C_1 \int_0^{\frac{x}{\sqrt{4kt}}} e^{-s^2} ds + C_2 \quad (**)$$

Use IC:

⑧

$$u(x,0) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Fix $x > 0$ and send $t \rightarrow 0$ in (**), so:

$$\frac{x}{\sqrt{4kt}} \rightarrow \infty$$

$$\text{Then: } 1 = u(x,0) = c_1 \int_0^{\infty} e^{-s^2} ds + c_2$$

$$\text{Fix } x < 0 \text{ and send } t \rightarrow 0, \frac{x}{\sqrt{4kt}} \rightarrow -\infty$$

$$\text{Then: } 0 = u(x,0) = c_1 \int_0^{-\infty} e^{-s^2} ds + c_2$$

key result:
(Gaussian integral)

$$\begin{aligned} \int_0^{\infty} e^{-s^2} ds &= \int_{-\infty}^0 e^{-s^2} ds = \frac{\sqrt{\pi}}{2} \\ &= \frac{1}{2} \int_{-\infty}^{\infty} e^{-s^2} ds \end{aligned}$$

IC eqs become:

(9)

$$\begin{aligned} c_1 \frac{\sqrt{\pi}}{2} + c_2 &= 1 \\ -c_1 \frac{\sqrt{\pi}}{2} + c_2 &= 0 \end{aligned} \Rightarrow \begin{aligned} c_2 &= \frac{1}{2} \\ c_1 &= \frac{1}{\sqrt{\pi}} \end{aligned}$$

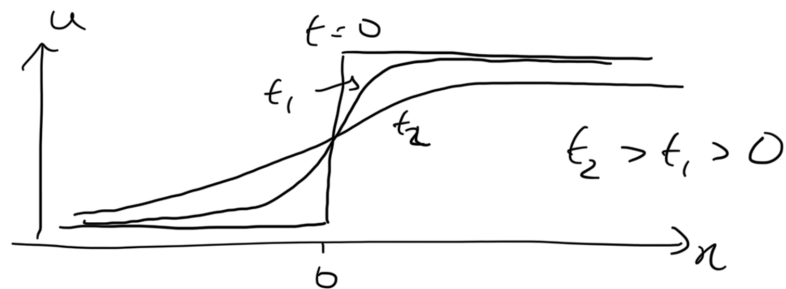
$$\Rightarrow u(x, t) = \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-s^2} ds + \frac{1}{2}$$

The 'error function' $\text{erf}(z)$ is defined by:

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2} ds$$

$$\Rightarrow u(x, t) = \frac{1}{2} \left(1 + \text{erf}\left(\frac{x}{\sqrt{4kt}}\right) \right)$$

Sketch:



How do we use this solⁿ to solve the general Cauchy problem? ⑩

$$u_t - ku_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0$$

$$u(x, 0) = \phi(x)$$

u bounded as $x \rightarrow \pm \infty$

Lemma: Suppose $v(x, t)$ solves the heat eq. Then the following are solⁿs

- $v(x-y, t)$, for any y
- $a_1 v(x-y_1, t) + a_2 v(x-y_2, t)$, for any a_1, a_2, y_1, y_2
- $\int_a^b v(x-y, t) \psi(y) dy$ for any a, b , function ψ
- $v_\alpha(x, t)$

Let $F(x, t)$ be the solⁿ with step function IC.

So: $F(x,t) = \frac{1}{2} \left(1 + \operatorname{erf} \left(\frac{x}{\sqrt{4kt}} \right) \right)$

(11)

Goal: use above moves to transform $F(x,t)$ into a solⁿ of (*) with IC $\phi(x)$

Solution of
 $u_t - ku_{xx} = 0$

IC ($t=0$)

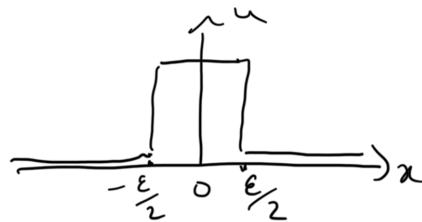
$F(x,t)$



$F(x+\epsilon, t)$



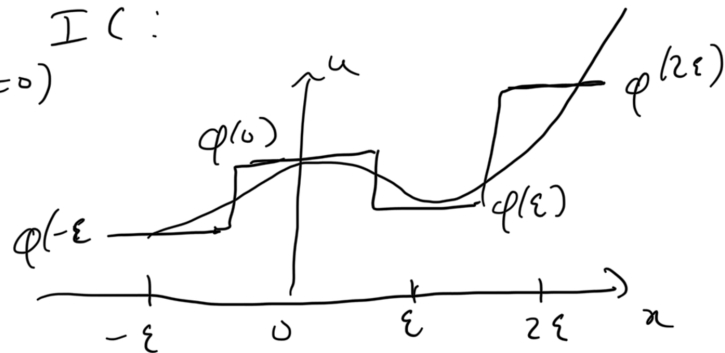
$F(x+\frac{\epsilon}{2}, t) - F(x-\frac{\epsilon}{2}, t)$



$$\sum_{n \in \mathbb{Z}} \phi(n\epsilon) \cdot \left[F(x-n\epsilon+\frac{\epsilon}{2}, t) - F(x-n\epsilon-\frac{\epsilon}{2}, t) \right]$$

Has $IC :$
($t=0$)

(12)



- Bumps at points $n\epsilon$ with width ϵ , and height $\phi(n\epsilon)$.
- As $\epsilon \rightarrow 0$ the IC approaches $\phi(x)$ and the sum becomes an integral:

$$SoIC: \sum_{n \in \mathbb{Z}} \phi(n\epsilon) \cdot \underbrace{\left[F(x - n\epsilon + \frac{\epsilon}{2}, t) - F(x - n\epsilon - \frac{\epsilon}{2}, t) \right]}$$

$$\approx \epsilon F_n(x - n\epsilon, t)$$

$$\rightarrow \int_{-\infty}^{+\infty} F_n(x-y, t) \phi(y) dy \quad \text{as } \epsilon \rightarrow 0$$