

Lecture 7

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Recap: Key takeaways so far

- Characteristics (Logan 1.2)
 - Solve 1st order PDEs
- Conservation laws/PDE derivations
 - Where do PDEs come from, how to model phenomena
 - Heat/diffusion, wave, Laplace eqs
- Special solⁿs
 - Steady state, standing waves

Next:- Classify PDEs

- Solve heat, wave eq.s on unbounded domains

Recap:

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Last time, we introduced the
Laplacian, ∇^2 , which acts on spatial variables
e.g. for $u = u(x, y)$ or $u(x, y, t)$

$$\text{we have } \nabla^2 u = u_{xx} + u_{yy}$$

Laplace's eq. for $u = u(x, y)$:

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

- Functions which satisfy $\nabla^2 u = 0$ are called harmonic.

e.g. $u(x, y) = x + y, xy, x^2 - y^2$
are all harmonic.

- Laplace's eq. comes with a domain, $\Omega \subseteq \mathbb{R}^2$:

$$\nabla^2 u = 0 \quad \text{for } (x, y) \in \Omega$$

Ω could be the whole plane, i.e. $\Omega = \mathbb{R}^2$

- Laplace's eq. comes with BCs:

Dirichlet BCs: specify u on the bdry, $\partial\Omega$

Neumann BCs: specify $\underline{n} \cdot \nabla u$ on $\partial\Omega$

where $\underline{n}$ is the outward unit
normal to $\partial\Omega$:

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We will see examples of this later in the class

Classification of PDE (Logan 1.9)

We have 3 main PDEs for functions
of 2 variables:

1) Heat/diffusion: $u = u(x, t)$

$$u_t - k u_{xx} = 0, \quad k > 0$$

2) Wave eq: $u = u(x, t)$

$$u_{tt} - c^2 u_{xx} = 0$$

3) Laplace eq: $u = u(x, y)$

$$u_{xx} + u_{yy} = 0$$

Remarks:

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- 1) & 2) are evolution eqs which specify how u changes in time. Typically we prescribe an initial state, along with BCs and ask for the state at time t .
- We think of 3) as a static problem
- Choosing t to be a 'time' variable in 1) & 2) but not 3) is not as arbitrary as it seems! (Next lecture)

Classification

Let $u(x, y)$ be a function of 2 vars.
Here we think of x, y as general variables,
not necessarily space or time vars)
Consider 2nd order PDEs of the form:

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$$A u_{xx} + B u_{xy} + C u_{yy} + F(x, y, u, u_x, u_y) = 0$$

where A, B, C are constants and F is some function.

The discriminant of this PDE is:

$$D = B^2 - 4AC$$

A PDE is: hyperbolic if $D > 0$
parabolic if $D = 0$
elliptic if $D < 0$

E.g:

1) Heat eq: $u_t - k u_{xx} = 0$, $k > 0$

Writing y for t and comparing with above we get:

$$A = -k, B = 0, C = 0$$

$$\Rightarrow D = B^2 - 4AC = 0$$

$\Rightarrow$ Heat eq. is parabolic

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2) Wave eq: $u_{tt} - c^2 u_{xx} = 0$

set $y = t$, gives:

$$A = -c^2, B = 0, C = 1$$

$$\Rightarrow D = B^2 - 4AC = 4c^2 > 0$$

$\Rightarrow$ wave eq. is hyperbolic

3) Laplace eq: $u_{xx} + u_{yy} = 0$

$$\Rightarrow A = 1, C = 1$$

$$\Rightarrow D = B^2 - 4AC = -4 < 0$$

$\Rightarrow$ Laplace eq. is elliptic

Remarks:

- PDEs of the same class behave similarly, e.g. hyperbolic eqs are wave-like
- Terminology comes from conic sections

- In general, the coefficients A, B, C could be functions x, y , in which case the PDE class would vary over the plane. ⑦

Next: Having set up our 3 main PDEs we will show how to solve them. Plan:

- ① 1st order PDE (characteristics) ✓
- ② Intro to heat, wave, Laplace eqs ✓
- ③ Unbounded domains •
- ④ Bounded domains

③ Unbounded domains (Logan Ch 2)

- Infinite & semi-infinite domains are easier because the BCs are looser. We start with the heat eq.

Cauchy problem for heat eq. (Logan 2.1) (8)

Consider the heat eq. for $u = u(x, t)$
on the real line:

$$u_t = k u_{xx}, \quad k > 0$$

Domain: $x \in \mathbb{R}, t > 0$ (*)

IC: $u(x, 0) = \phi(x), x \in \mathbb{R}$

'BC's': For infinite domains, we typically
also have some decay conditions.

We take $u_x \rightarrow 0$ as $x \rightarrow \pm \infty$

Rmk: This is a pure initial value
problem which is also called a
Cauchy problem. How to solve it?

Given a solution to the heat eq. ⑨
we can generate more solⁿs.

e.g. suppose $G(x, t)$ solves the heat eq.

$$\text{so } G_t - k G_{xx} = (\partial_t - k \partial_x^2) G = 0$$

Then:

$$\begin{aligned} (\partial_t - k \partial_x^2) G_x &= G_{xt} - k G_{xxx} \\ &= (G_t - k G_{xx})_x = 0 \end{aligned}$$

$\Rightarrow G_x$ solves heat eq.

$$\begin{aligned} \text{and: } (\partial_t - k \partial_x^2) G(x-y, t) &= 0 \\ \Rightarrow G(x-y, t) &\text{ solves heat eq.} \end{aligned}$$

and: for any ψ :

$$(\partial_t - k \partial_x^2) \int_{-\infty}^{+\infty} G(x-y, t) \psi(y) dy$$

$$= \int_{-\infty}^{\infty} (\partial_t - k \partial_x^2) G(x-y, t) \varphi(y) dy \quad (10)$$

$$= 0$$

So $\int_{-\infty}^{\infty} G(x-y, t) \varphi(y) dy$ solves the heat eq.

We will first solve the special case where φ is a step function, and then use the above moves to generate a general solution.

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For step-function ϕ we have:

$$u_t - k u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0$$

$$u(x, 0) = \begin{cases} \alpha, & x > 0 \\ 0, & x < 0 \end{cases}$$

How to solve this?

Dimensional analysis - a magic trick

The variables and constants in the problem are: x, t, u, k, α

Interpret u as temperature, x as length and t as time.

Write $[\cdot]$ for dimensions:

$$[u] = \text{temp.}, \quad [x] = \text{length}, \quad [t] = \text{time}$$

$$\Rightarrow [u_t] = \frac{\text{temp.}}{\text{time}}, \quad [u_{xx}] = \frac{\text{temp.}}{\text{length}^2}$$

$$\Rightarrow [k] = [u_t / u_{xx}] = \frac{\text{length}^2}{\text{time}}$$

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and $[\alpha] = [u] = \text{temp.}$

There are 2 dimensionless quantities:

$$\frac{1}{\alpha} u \quad \text{from temp. params}$$

$$\frac{x}{\sqrt{4kt}} \quad \text{from length, time params}$$

[Factor of 4 is for convenience]

Since $\frac{1}{\alpha} u$ is dimensionless, it must be a function of dimensionless quantities

$$\Rightarrow \frac{1}{\alpha} u = f\left(\frac{x}{\sqrt{4kt}}\right)$$

for some f . For simplicity, we now set $\alpha = 1$, and let $y = \frac{x}{\sqrt{4kt}}$, so:

$$u(x,t) = f(z(x,t))$$

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We find f by substituting this expression into the heat eq: $u_t - ku_{xx} = 0$:

$$u_t = f'(z) \frac{\partial z}{\partial t}$$

$$= -\frac{1}{2} \cdot \frac{x}{\sqrt{4kt^3}} f'(z) = \frac{-\eta}{2t} f'(z)$$

$$u_x = f'(z) \cdot \frac{\partial z}{\partial x}$$

$$= \frac{1}{\sqrt{4kt}} f'(z)$$

$$u_{xx} = \frac{1}{4kt} f''(z)$$

Then $u_t - ku_{xx} = 0$

$$\Rightarrow \frac{-\eta}{2t} f' - \frac{1}{4kt} f'' = 0$$

$$\Rightarrow f'' + 2z f' = 0$$