

## Lecture 6

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Recap: (Hws 1 & 2, practice midterm)

- 1st order PDE - characteristics,  
change of co-ords
- 2nd order: (1 space dim)
  - Diffusion  $u_t = D u_{xx}$   
/ heat (+ advection, decay, etc.)
    - Steady state sol<sup>n</sup>s
  - Wave eq:  $u_{tt} = c^2 u_{xx}$ 
    - Standing waves
  - BCs / ICs
    - Dirichlet ( $u$  known on bdry)
    - Neumann ( $u_x$  known on bdry)
- Midterm 1: up to (including) lecture 5  
(+ ODEs, separation of variables,  
homog. sol<sup>n</sup> + particular integral, etc)

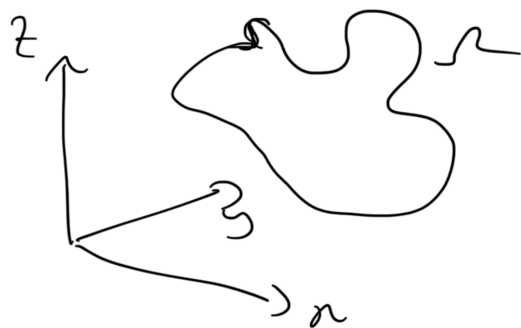
## Higher dimensions (Lec 1.7)

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- Most physical processes occur in more than 1 spatial dimension
- We can generalize 1d derivations using vector calculus

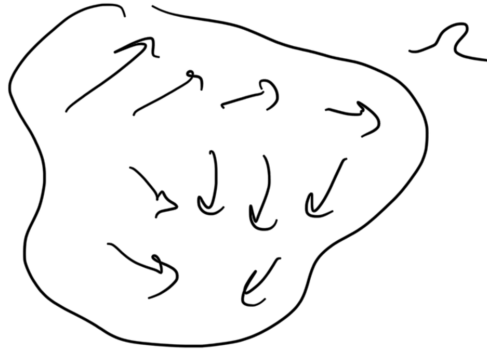
### Heat flow in 2d/3d

- 2d & 3d are similar - we work in 3d
- Consider heat flow in domain  $\Omega \subseteq \mathbb{R}^3$



- Let  $u(x, y, z, t)$  be temp. in  $\Omega$  at time  $t$ , and let  $c_u$  be the thermal energy density

- Let  $\phi(x, y, z, t)$  be the energy flux (3) vector, i.e.  $\phi$  is the vector field of heat flow



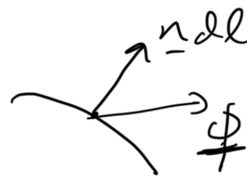
- The rate of heat flow across a surface element  $\underline{dS}$  is  $\phi \cdot \underline{dS}$



$$\underline{dS} = \underline{n} dA$$

$\uparrow$                        $\uparrow$   
 normal                  area  
                                 element

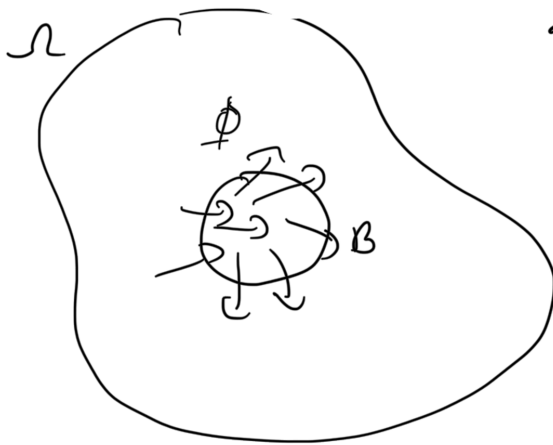
- In 2d, flow across a line element is  $\phi \cdot \underline{n} dl$



- Let  $f(x, y, z, t)$  be the rate at which heat is being produced.

Consider thermal energy in a subdomain

$B \subseteq \Omega$ :



- Energy flows according  $\phi$
- Created by  $f$

Conservation / Balance

$$\frac{d}{dt} \int_B c u \, dV = - \int_{\partial B} \phi \cdot \underline{dS} + \int_B f \, dV$$

Change in energy = flux out + energy sources/sinks  
( $\partial B$  is bdry)

Rmk: This applies to any quantity (5) that has a spatially varying density

How to get PDE? Vector calculus

Divergence thm:

Let  $\underline{F}$  be a (smooth) vector field in a region  $B \subseteq \mathbb{R}^3$ . Then:

$$\int_{\partial B} \underline{F} \cdot d\underline{S} = \int_B \nabla \cdot \underline{F} dV$$

flux out = total divergence  
in domain

where  $\underline{F} = (F_1, F_2, F_3)$

and:  $\nabla \cdot \underline{F} = \partial_x F_1 + \partial_y F_2 + \partial_z F_3$

Apply to conservation law:

$$\frac{d}{dt} \int_B c u \, dV = - \int_{\partial B} \phi \cdot \underline{dS} + \int_B f \, dV \quad (6)$$

$$\Rightarrow \int_B c u_t \, dV = - \int_B \nabla \cdot \phi \, dV + \int_B f \, dV$$

True for any  $B \subseteq \Omega$

$$\begin{aligned} \Rightarrow c u_t &= -\nabla \cdot \phi + f \\ &= -\partial_n \phi_1 - \partial_2 \phi_2 - \partial_3 \phi_3 + f \end{aligned}$$

To get a PDE we need a relationship between  $\phi$  and  $u$ . In 1 space dim

we had  $u = u(x, t)$ ,  $\phi = \phi(x, t)$

and  $\phi = -K u_x$  (constant  $K > 0$ )

(in 1 space dim  $\phi$  is effectively a scalar)

Fick's Law in higher dim:

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$$\begin{aligned}\underline{\phi} &= -\kappa \nabla u \quad (\text{const. } \kappa > 0) \\ &\quad \uparrow \\ &= (u_x, u_y, u_z)\end{aligned}$$

This gives PDE:

$$\begin{aligned}C u_t &= -\nabla \cdot \underline{\phi} + f \\ &= \kappa \nabla \cdot (\nabla u) + f\end{aligned}$$

$$\begin{aligned}\text{Where } \nabla \cdot (\nabla u) &= \nabla \cdot (u_x, u_y, u_z) \\ &= u_{xx} + u_{yy} + u_{zz} \\ &:= \nabla^2 u \quad (\text{Laplacian of } u)\end{aligned}$$

This gives PDE:

$$C u_t = \kappa \nabla^2 u + f \quad \text{for } (x, y, z) \in \Omega$$

Def<sup>n</sup>: The Laplacian

Let  $u$  be a scalar function ⑧  
 depends on space variables and  
 possibly also time. The Laplacian of  $u$ ,  
 $\nabla^2 u$  is the sum of 2nd derivs w.r.t.  
 space variables.

e.g. for  $u = u(x, y, z, t)$ , (3d)

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$$

for  $u = u(x, y)$ , (2d)

$$\nabla^2 u = u_{xx} + u_{yy}$$

for  $u = u(x, t)$  or  $u = u(x)$

$$\nabla^2 u = u_{xx} \quad (1d)$$

Remarks:

- The Laplacian  $\nabla^2$  is a linear differential operator
- We've only defined  $\nabla^2$  for scalar fields. It can be defined for

vector fields as well but is 9  
more involved

- Laplacian of  $u$  is sometimes denoted  $\Delta u$ , e.g.  $\nabla^2 u = \Delta u$

- Relationship with div, grad:

$$\nabla^2 u = \nabla \cdot (\nabla u)$$

- Heat / diffusion eq. in higher dim:

$$u_t = D \nabla^2 u, \quad \text{const. } D > 0$$

for  $u = u(x, y, z, t)$  or  $u = u(x, y, t)$

and  $(x, y, z) \in \Omega$  (some domain)

Heat eq. with source:

$$u_t = D \nabla^2 u + f$$

for  $f = f(x, y, z, t)$  or  $f(x, y, t)$

- The time independent cases also

give important eqs:

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Poisson's eq:

$$\nabla^2 u = -f \quad \text{in } \Omega$$

Laplace's eq:

$$\nabla^2 u = 0 \quad \text{in } \Omega$$

2nd order in time gives

Wave eq:

$$u_{tt} = c^2 \nabla^2 u \quad \text{on } \Omega$$

Rmk: These eqs are fundamental for describing dynamics of spatially varying quantities in 2d/3d, e.g. fluid flow, vibrations, quantum mechanics etc.

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## Boundary Conditions

Just like in 1d, PDEs in higher dim. need BCs

Consider heat eq.  $u_t = D \nabla^2 u$

1d:



$$\mathcal{R} = [0, L] \subset \mathbb{R}$$

Dirichlet BCs:  $u(0, t) = g_1(t)$

$$u(L, t) = g_2(t)$$

e.g.  $u$  known on bdry

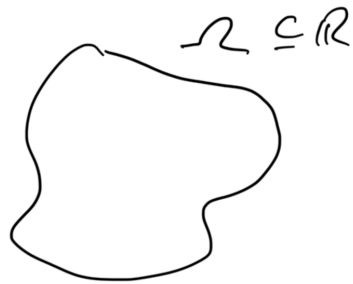
Neumann BCs:  $u_x(0, t) = g_1(t)$

$$u_x(L, t) = g_2(t)$$

e.g.  $u_x$  known on bdry

2d:

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$\partial\Omega = \text{bdry of } \Omega$   
(curve)

Dirichlet BC:  $u(x, y, t) = g(x, y, t)$

for  $(x, y) \in \partial\Omega$  (bdry)

also written as:

$$u|_{\partial\Omega} = g$$

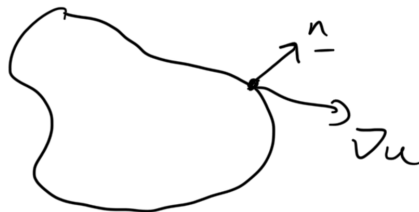
i.e.  $u$  is known on bdry

Neumann BC:

$$\underline{n}(x, y) \cdot \nabla u(x, y, t) = g(x, y, t)$$

for  $(x, y) \in \partial\Omega$ , where  $\underline{n}(x, y)$

is unit normal to  $\partial\Omega$  at  $(x, y)$



Write as:

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$$\underline{n} \cdot \nabla u \Big|_{\partial\Omega} = g \quad \text{on } \partial\Omega$$

$$\text{or } \frac{\partial u}{\partial n} \Big|_{\partial\Omega} = g$$

Neumann BC  $\rightarrow$  flux out of domain,  
 $\underline{n} \cdot \nabla u$  is known.

Rmk: - PDE notation is 2d, 3d  
is hard!

• Helps to think pictorially

E.g. Vibrating drum

A drum occupies region  $\Omega \subseteq \mathbb{R}^2$   
 $u(x, y, t)$  is the height of the drum

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The drum is clamped on its bdry  
 so  $u|_{\partial\Omega} = 0$  (Dirichlet BC)

The height of the drum obeys the  
 wave eq:

$$u_{tt} = c^2 \nabla^2 u \quad \text{on } \Omega$$

$$u|_{\partial\Omega} = 0$$

where  $c^2 = \frac{T}{\rho} = \frac{\text{tension of drum}}{\text{density}}$

• Can derive this eq. by following  
 same steps as for the vibrating  
 string