

Lecture 5

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Recap:

Diffusion: $u_t = D u_{xx}$, $D > 0$

Wave eq: $u_{tt} = c^2 u_{xx}$, c is speed

Vibrations of a string



* String fixed at $x=0$, $x=l$

$u(x, t)$ is the height of the string at time t , position x .

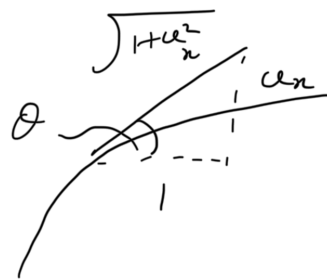
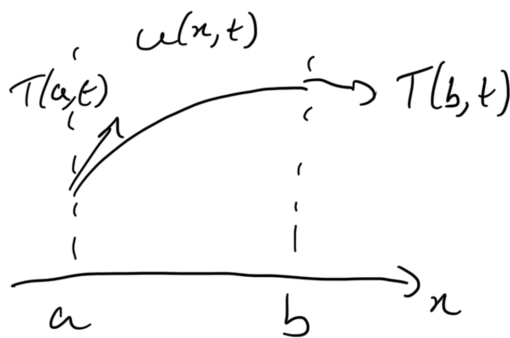
• Density ρ (= mass/length)

Assumptions:

• Small deflections, i.e. $|u_x|$ is small

- Only force acting on string is tension (no gravity, damping etc.) ②
- No horizontal motion of string (i.e. only vertical)

Let $T(x,t)$ be the tension. T acts tangent to the string:



- $\theta(x,t)$ is the angle of the tangent to the string, so:

$$\cos \theta = \frac{1}{\sqrt{1+u_x^2}}, \quad \sin \theta = \frac{u_x}{\sqrt{1+u_x^2}}$$

- T is tension force along the string

⇒ Total force on string in region $[a, b]$ ③
is:

horiz: $T \cos \theta \Big|_a^b = \frac{T}{\sqrt{1+u_n^2}} \Big|_a^b$

vert: $T \sin \theta \Big|_a^b = \frac{T u_n}{\sqrt{1+u_n^2}} \Big|_a^b$

We can simplify this using the small deflection assumption, $|u_n| \ll 1$

Taylor series: $(1+u_n^2)^{1/2} = 1 + \frac{1}{2}u_n^2 - \frac{1}{8}u_n^4 + \dots$
 $\uparrow \quad \uparrow$
 each term gets smaller

≈ 1

So the forces become:

horiz: $T \cos \theta \Big|_a^b \approx T \Big|_a^b = \int_a^b T_n dx$

vert: $T \sin \theta \Big|_a^b \approx T u_n \Big|_a^b = \int_a^b (T u_n)_x dx$

The accelerations are:

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horiz: 0 (by assumption)

$$\text{vert: } \int_a^b u_{tt} dx$$

Newton's 2nd law " $F=ma$ ":

$$\int_a^b T_x dx = 0 \quad (\text{horiz.})$$

$$\int_a^b (Tu_x)_x dx = \int_a^b \rho u_{tt} dx \quad (\text{vert.})$$

Holds for all intervals $[a, b]$

$$\Rightarrow T_x = 0 \text{ so } T(x, t) = T(t)$$

$$\text{and } (Tu_x)_x = Tu_{xx}$$

Vertical eq:

$$Tu_{xx} = \rho u_{tt}$$

$$\Rightarrow u_{tt} = c^2 u_{xx} \text{ for } c = \sqrt{\frac{T}{\rho}} \quad (5)$$

Rmk:

- Typically assume $T(t)$ is also time-indep so $T = \text{const.}$ and speed $c = \sqrt{\frac{T}{\rho}}$ is const.
- For const. c the general solution is $u(x,t) = \overset{\uparrow}{F(x-ct)} + \overset{\uparrow}{G(x+ct)}$
right-moving wave
left moving

(Will show this later)

- For a plucked string, our BCs are: $u(x,0) = u(l,t) = 0$

To get a specific solution, we need 2 ICs, since the eq. is 2nd order in time. E.g. we need

Initial position: $u(x, 0) = f(x)$ (6)
 Initial velocity: $u_t(x, 0) = g(x)$
 for $0 < x < l$

• There are many variations, e.g.:

- Air resistance term, ru_t :

$$u_{tt} - c^2 u_{xx} + ru_t = 0$$

- External force, $f(x, t)$:

$$u_{tt} - c^2 u_{xx} = f(x, t)$$

Standing waves

Consider a vibrating string:

$$u_{tt} = c^2 u_{xx}$$

BCs: $u(0, t) = u(l, t) = 0$



• Standing waves are solⁿs of the

form $u(x,t) = \phi(x) \cos \omega t$

⑦

(or $\phi(x) \sin \omega t$ or in
general $\phi(x) e^{i\omega t}$)

- Each part of the string oscillates with:
 - Same freq. ω , in phase
 - Different amplitude $\phi(x)$
- Solutions only exist for certain ω :

Substitute $u = \phi(x) \cos \omega t$ in to wave eq:

$$-\omega^2 \phi \cos \omega t = c^2 \phi'' \cos \omega t$$

$$\text{with } \phi(0) \cos \omega t = \phi(l) \cos \omega t = 0 \quad \forall t$$

$\Rightarrow$ ODE BVP:

$$\phi'' = -\frac{\omega^2}{c^2} \phi, \quad \phi(0) = \phi(l) = 0$$

General solⁿ:

$$\phi(x) = A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c}$$

$$\text{BC 1: } \phi(0) = A = 0$$

$$\Rightarrow \phi(x) = B \sin \frac{\omega x}{c}$$

$$BC2: \phi(l) = B \sin \frac{\omega l}{c} \quad (8)$$

$$\Rightarrow B = 0 \quad \text{or} \quad \sin \frac{\omega l}{c} = 0$$

If $B = 0$ then $\phi(x) = 0$ and $u(x, t) = 0$

We are interested in non-zero solⁿs

$$\Rightarrow \text{we need } \sin \frac{\omega l}{c} = 0$$

Note: l and c are fixed by the problem, ω is the frequency of the standing wave. The condition $\sin \frac{\omega l}{c} = 0$ means: the allowed frequencies ω of a standing wave satisfy $\sin \frac{\omega l}{c} = 0$

$$\Rightarrow \frac{\omega l}{c} = n\pi \quad \text{for } n \in \mathbb{Z}$$

$$\text{Allowed frequencies: } \omega_n = \frac{n\pi c}{l}$$

Modes: $\phi_n(x) = \sin \omega_n x$ ⑨
 $= \sin\left(\frac{n\pi x}{\ell}\right), \quad n \in \mathbb{Z}$

- $B \phi_n(x)$ satisfies the BVP for any constant B , modes defined up to a multiplicative constant.
- $n=0$ gives $\phi_n=0$ so is trivial
- $n<0$ gives the same mode as $n>0$ up to a sign: $\phi_n(x) = -\phi_{-n}(x)$
- We can take the modes to have $n>0$:

$$\phi_n(x) = \sin\left(\frac{n\pi x}{\ell}\right), \quad n=1, 2, 3, \dots$$

The standing waves are:

$$\begin{aligned} u_n(x, t) &= \cos\left(\frac{n\pi c t}{\ell}\right) \sin\left(\frac{n\pi x}{\ell}\right) \\ &= \cos(\omega_n t) \phi_n(x), \quad n=1, 2, 3, \dots \end{aligned}$$

Rmks:

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- The BVP problem above is an eigenvalue problem. The frequencies ω_n are the c -values and modes ϕ_n are eigenfunctions
- Eigenfunctions ϕ_n are defined up to a constant multiplier, like eigenvectors
- The frequencies ω_n are the fundamental frequencies of the string

Quantum Mechanics (Logan 1.6)

Consider a particle in 1d. Classically, the particle has a position $x \in \mathbb{R}$. In QM, the state of the particle is a function $\Psi(x, t) \in \mathbb{C}$ called the wavefunction. The

probability density of finding the particle (11)
at x is $|\Psi(x, t)|^2$.

Ψ obeys the Schrödinger equation: (SE)

$$i\hbar \Psi_t = -\frac{\hbar^2}{2m} \Psi_{xx} \quad (\text{free particle})$$

where $\hbar$ is Planck's constant

m is particle mass

If the 'energy cost' of being at position x is $V(x)$, then the SE is:

$$i\hbar \Psi_t = -\frac{\hbar^2}{2m} \Psi_{xx} + V(x) \Psi$$

- The particle is typically more likely to be at places where $V(x)$ is lower

- Probability is conserved:

$$\int_{-\infty}^{\infty} |\Psi|^2 dx = 1 \quad \forall t$$

- Looking for solutions of the form (12)
 $\Psi(x,t) = e^{-iEt/\hbar} \cdot \phi(x)$ gives
 the time-indep SE:

$$E\phi = -\frac{\hbar^2}{2m} \phi'' + V(x)\phi$$

- E has the interpretation of 'energy'

For a particle in a '1d box', $0 < x < l$
 we have:

$$E\phi = -\frac{\hbar^2}{2m} \phi'', \quad 0 < x < l$$

$$\text{BCs: } \phi(0) = \phi(l) = 0$$

$\Rightarrow$ standing wave solⁿs.

Higher dimensions (Logan 1.7)

Most physical processes occur in more
 than 1 spatial dimension.