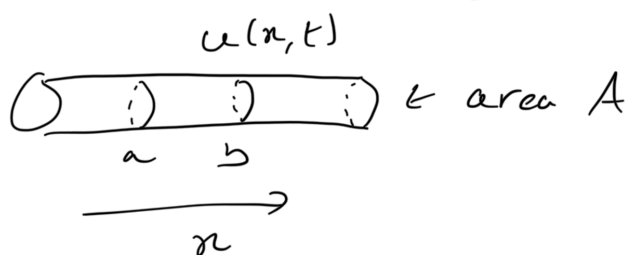


Lecture 4

①

Recap:

- Conservation laws: (Logan 1.2)



Quantity: $u(x, t)$

Flux: $\phi(x, t)$ (movement of u)

Sources: $f(x, t)$ (creation of u)

Balance in domain $[a, b]$:

$$\frac{d}{dt} \int_a^b A u(x, t) dx = A \phi(a, t) - A \phi(b, t) + \int_a^b A f(x, t) dx$$

$$\Rightarrow \int_a^b u_t dx = - \int_a^b \phi_x dx + \int_a^b f dx$$

②

$$\Rightarrow \int_a^b (u_t + \phi_x - f) dx = 0$$

Holds for any domain $[a, b]$

$$\Rightarrow u_t + \phi_x = f$$

Next: origin of PDEs

→ diffusion, waves, Laplace (Logan 1.3, 1.5, 1.7-1.9)

Diffusion eq (Logan 1.3)

Suppose $u(x, t)$ is e.g. density / concentration of substance

Fick's law for flux:

$$\phi(x, t) = -D u_x, \text{ const. } D > 0$$

⇒ • movement from high u to low u

• Steeper gradient gives higher flux

Conservation / balance law:

$$u_t + \phi_x = 0$$

$$\Rightarrow \boxed{u_t = D u_{xx}}$$

③

diffusion eq. (or heat eq.)

D is called the diffusivity
(or thermal conductivity for heat eq.)

Remarks

- Some physical setting have non-constant diffusivity, $D(x)$

$$\text{flux: } \phi = -D(x)u_x$$

$$\text{PDE: } u_t = (Du_x)_x = D'u_x + Du_{xx}$$

- Can also find nonlinear diffusion eqs where $D = D(u)$:

$$u_t = (D(u)u_x)_x$$

- Combine with other terms, e.g.
diffusion + advection:

$$u_t + cu_x = Du_{xx}$$

(4)

- Dimensions:

Suppose $u(x, t)$ is a temperature

$$[u] = \text{temp.}$$

Then: $[u_t] = \frac{\text{temp.}}{\text{time}}$

$$[u_{xx}] = \frac{\text{temp.}}{\text{length}^2}$$

$$\text{So } u_t = D u_{xx} \Rightarrow [D] = \frac{\text{length}^2}{\text{time}}$$

Consider heat flow in a region of length L . The timescale T for temperature change is:

$$D = \frac{L^2}{T} \Rightarrow T = \frac{L^2}{D}$$

- Dimensions give rough estimates for heat flow

Boundary / initial conditions

⑤

Diffusion eqs often come with initial conditions (IC) & Boundary conditions (BCs)

Eg: heat flow in 1d domain $0 < x < l$

IC has form:

$$u(x, 0) = u_0(x), \quad 0 < x < l$$

$u_0(x)$ is initial temperature profile

There are 3 types of common BC:

1) Dirichlet BC:

- Specify u at the boundary

$$u(0, t) = g_1(t), \quad u(l, t) = g_2(t)$$

where g_1, g_2 are given

i.e. u is known at boundary

2) Neumann BC:

- Specify u_n at boundary:

(6)

$$u_n(0,t) = h_1(t), \quad u_n(l,t) = h_2(t)$$

where h_1, h_2 are given

i.e. flux of u is known at boundary

For 1) & 2), BC is homogeneous if $g, h = 0$

3) Robin BC:

- Combination of u, u_n given at boundary

$$a u(0,t) + b u_n(0,t) = \psi_1(t)$$

$$a u(l,t) + b u_n(l,t) = \psi_2(t)$$

where $\psi_1(t), \psi_2(t)$ are given

e.g. Newton's law of cooling states:

heat flux $\propto$ temperature difference
between bar & environment

$$\Rightarrow \underset{\substack{\uparrow \\ \text{flux}}}{b u_n(0,t)} = \underset{\substack{\uparrow \\ \text{bar}}}{a u(0,t)} - \underset{\substack{\uparrow \\ \text{environment}}}{\psi_1(t)}$$

for some a, b

Notes:

⑦

- A well-posed problem is a PDE accompanied by appropriate ICs & BCs so that it has a unique solⁿ with a correct physical meaning
- The above terminology for BCs applies to general PDEs (not just diffusion)

Steady State solutions

- Many time-dependent PDEs have solⁿs which approach steady states $u = u(x)$ which are time-independent

e.g. Diffusion/heat eq:

$$u_t = D u_{xx}$$

Steady state solⁿ $u = u(x)$ has ⑧

$u_t = 0$ so :

$$Du'' = 0 \quad (\text{using } u' = u_x)$$

→ 2nd order ODE

Integrate twice:

$$u(x) = ax + b$$

for consts a, b set by B.Cs.

Eg: Diffusion-decay eq:

$$u_t = Du_{xx} - ru, \quad 0 < x < L, \quad t > 0$$

for consts $D, r > 0$

[u has flux $\phi = -Du_x$ and is
destroyed at rate ru]

Take B.Cs:

$$u(0, t) = 0, \quad -Du_x(L, t) = -1, \quad t > 0$$

and IC:

⑨

$$u(x, 0) = g(x), \quad 0 < x < L$$

Rmk: This model could describe
diffusing pollutant (Du_{xx} term) destroyed
at rate r ($-ru$ term)



At 0: $u(0, t) = 0$, u held at 0

At L : $-Du_x(L, t) = -1 \Rightarrow$ const. flux
of u into domain

Steady-state problem:

$$u = u(x), \text{ so } u_t = 0,$$

$\Rightarrow$ Boundary value problem: (BVP)

$$Du'' - ru = 0, \quad 0 < x < L$$

$$\text{BCs: } u(0) = 0, \quad -Du'(L) = -1$$

(10)

Ignore IC $\rightarrow$ assume transients have decayed and no longer affect solⁿ.

Solⁿ of BVP:

$$u'' = \frac{r}{D} u, \quad u(0) = 0, \quad u'(L) = \frac{1}{D}$$

general solⁿ:

$$u = A \sinh\left(x \sqrt{\frac{r}{D}}\right) + B \cosh\left(x \sqrt{\frac{r}{D}}\right)$$

for consts A, B

$$\text{BC 1, } u(0) = 0 \Rightarrow B = 0,$$

$$u = A \sinh\left(x \sqrt{\frac{r}{D}}\right)$$

$$\text{BC 2, } u'(L) = \frac{1}{D} \Rightarrow A \cdot \sqrt{\frac{r}{D}} \cosh\left(L \sqrt{\frac{r}{D}}\right) = \frac{1}{D}$$

⇒ steady state solⁿ:

(11)

$$u(x) = \frac{\sinh\left(x \sqrt{\frac{r}{D}}\right)}{\sqrt{rD} \cosh\left(2 \sqrt{\frac{r}{D}}\right)}$$

Remarks:

- An ODE BVP may not have a solⁿ.
Physically the PDE may not have a steady state solⁿ
- The steady state solⁿ could be unstable in general. Here it turns out that $u(x,t) \rightarrow$ steady state

Summary: diffusion in 1d, $u(x,t)$:

$$u_t = D u_{xx}, \quad D > 0$$

Can consider higher-dim (later)

Vibrations and waves

(12)

1d wave eq. for $u(x,t)$:

$$u_{tt} = c^2 u_{xx}$$

describes e.g. waves in a string

- c has dimensions length/time and is speed of wave

- 2nd order in time $\rightarrow$ waves

(contrast with first order in time $\rightarrow$ diffusion)

Vibrations of a taut string



String fixed at $x=0, x=l$

$u(x,t)$ is height of the string at

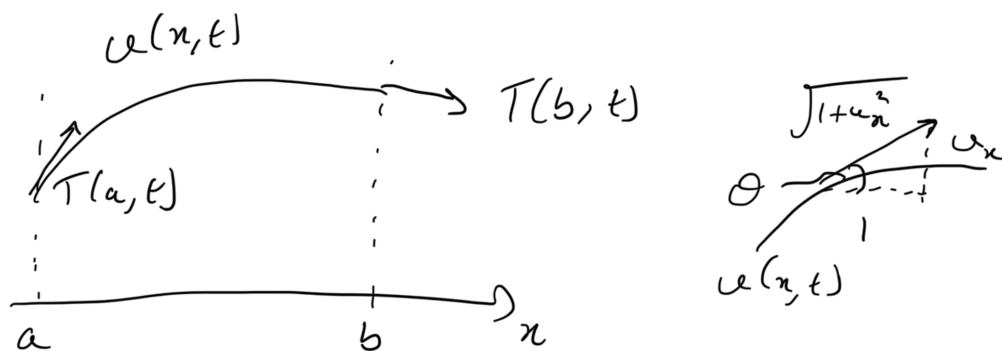
(13)

position x , time t .

Assumptions:

- Only force acting on string is tension (no gravity, damping etc.)
- No horizontal motion of string (i.e. only vertical)

Let $\tau(x, t)$ be the tension. T acts tangent to string:



- $\theta(x, t)$ is tangent angle to string so

$$\cos \theta = \frac{1}{\sqrt{1+u_x^2}}, \quad \sin \theta = \frac{u_x}{\sqrt{1+u_x^2}}$$

• T is tension force along the string (14)
 so in the forward direction.

$\Rightarrow$ Total force on string in region $[a, b]$ is:

horiz. $T \cos \theta = \left. \frac{T}{\sqrt{1+u_n^2}} \right|_a^b$

vertical: $T \sin \theta \Big|_a^b = \left. \frac{T u_n}{\sqrt{1+u_n^2}} \right|_a^b$

Equate accelerations:

horiz. accelⁿ: 0 (no horiz motion)

vertical accelⁿ: $\int_a^b u_{tt} \, dx$

Newton's 2nd law (" $F=ma$ "):