

Lecture 3

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Recap: Characteristics (Logan 1.2)

- Solve PDEs of form:

$$a(x, y) u_x + b(x, y) u_y = 0$$

- Char. curves $(x(\eta), y(\eta))$ satisfy:

$$\frac{dx}{d\eta} = a(x, y), \quad \frac{dy}{d\eta} = b(x, y)$$

• u is const. along chars

E.g:

$$\frac{1}{x^2} u_x + u_y = 0$$

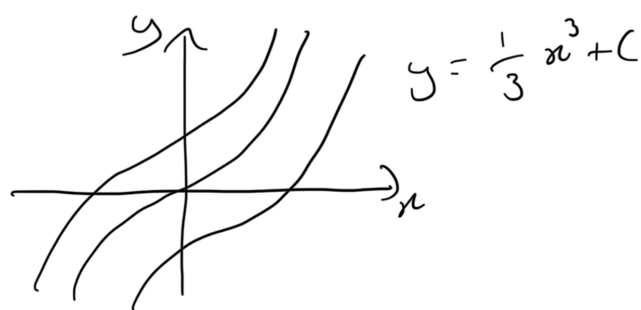
• Parametric eqs for chars:

$$\frac{dx}{d\eta} = \frac{1}{x^2}, \quad \frac{dy}{d\eta} = 1$$

- Method 1: divide: $\frac{dy}{dx} = x^2$

$$\Rightarrow y = \frac{1}{3} x^3 + C$$

• Sketch:



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• Const. C labels chars.

$\Rightarrow$ general solⁿ $u = f(C) = f(y - \frac{1}{3}x^3)$
for arbitrary function f .

Rmks: Doesn't give co-ordinate y along the chars.

• Method 2: Solve parametric eqs:

$$\frac{dx}{d\eta} = \frac{1}{x^2}, \quad \frac{dy}{d\eta} = 1$$

$$\Rightarrow \int x^2 dx = \eta + A \quad y = \eta + B$$

$$\Rightarrow \frac{1}{3}x^3 = \eta + A, \quad y = \eta + B$$

• Integration consts label each char. 'twice'.

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• Absorb into 1 const: $B=0, A=\xi$

$$\frac{1}{3}x^3 = \eta + \xi$$

Check that chars cover whole plane
i.e. no chars have been 'missed' by
e.g. sketching

• This gives our co-ord transformation:
 $\frac{1}{3}x^3 = \eta + \xi \Rightarrow x = (3\eta + 3\xi)^{\frac{1}{3}}$

$$y = \eta$$

• Transformed PDE: $u_\eta = 0$

$$\begin{aligned} \text{Check: } u_\eta &= u_x x_\eta + u_y y_\eta \\ &= u_x \cdot (3\eta + 3\xi)^{-\frac{2}{3}} + u_y \\ &= u_x \cdot \frac{1}{x^2} + u_y \end{aligned}$$

General solⁿ: $u = f(\xi) = f(\frac{1}{3}x^3 - y)$ (4)

Eg: Advection:

Consider function $u(x, t)$ describing a quantity u at position x and time t .

Advection eq:

$$u_t + cu_x = 0, \quad \text{const. } c > 0$$

• Chars: $(t(\eta), x(\eta))$ satisfy:

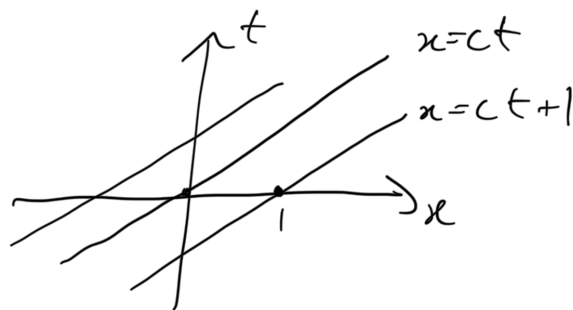
$$\frac{dt}{d\eta} = 1, \quad \frac{dx}{d\eta} = c$$

solve: $t = \eta + A, \quad x = c\eta + B, \quad \text{const. } A, B$

• Absorb into one const: $A=0, B=\xi$

$$\Rightarrow \begin{aligned} t &= \eta \\ x &= c\eta + \xi \end{aligned} \quad \text{so} \quad x = ct + \xi$$

• Sketch chars:

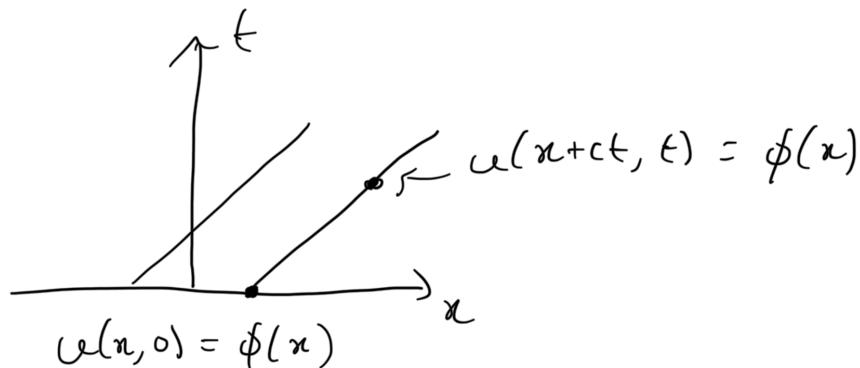


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→ Char's cover plane

- Transformed PDE: $u_\eta = u_t + cu_x = 0$
- General solⁿ: $u(x, t) = f(\xi) = f(x - ct)$
for arbitrary function f

- Suppose we have IC: $u(x, 0) = \phi(x)$
Then $u(x, 0) = f(x - c \cdot 0) = f(x) = \phi(x)$
 $\Rightarrow$ solⁿ is $u(x, t) = \phi(x - ct)$



- Solution flows along chars.

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Rmk:

- We can solve more complicated PDEs using chars.

e.g.: $u_t + cu_x = f(u)$ for some f

In co-ords: (ξ, η) with:

$$t = \eta$$

$$x = c\eta + \xi$$

we have: $u_\eta = f(u)$

we can solve this as an ODE with const. which depend on ξ . (HW1)

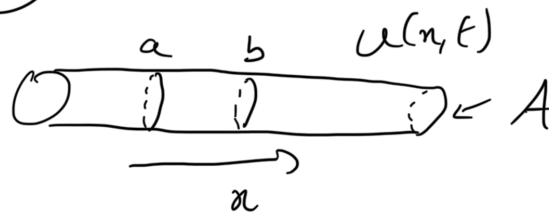
Conservation Laws (Logan 1.2)

For a conserved quantity:

$$\begin{aligned} \text{rate of change} &= \text{net flow} \\ \text{in domain} &\quad \text{into domain} \\ &\quad + \text{creation in domain} \end{aligned} \quad \textcircled{7}$$

• Eg animal population in fixed region
Quantitatively:

• Let $u(x, t)$ be the density of some quantity in a 1d domain



→ amount of quantity in section of width dx is $u(x, t) A dx$

• Let $\phi(x, t)$ be the flux → amount of quantity crossing section at x , time t

→ $A \cdot \phi(n, t)$ is amount of quantity ⑧
passing n at time t . (left to right)

• let $f(n, t)$ be the rate at which
the quantity is created or destroyed
at n , time t .

→ $f(n, t) A \, dn$ is amount of quantity
created in width dn per unit time

Balance:

change = flux + creation
in quantity
in section $[a, b]$

$$\Rightarrow \frac{d}{dt} \left(\int_a^b u(n, t) A \, dn \right) = A \phi(a, t) - A \phi(b, t) + \int_a^b f(n, t) A \, da$$

$$\Rightarrow \int_a^b u_t \cdot A dx = - \int_a^b \phi_x A dx + \int_a^b f A dx$$

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$$\Rightarrow \int_a^b (u_t + \phi_x - f) A dx = 0$$

$\rightarrow$ True for any domain $[a, b]$

$$\Rightarrow u_t + \phi_x = f$$

$\nearrow$ Change in quantity $\uparrow$ net flow $\nwarrow$ creation

Remarks:

- $\phi(x, t) > 0 \Rightarrow$ flow is left to right
 $\phi(x, t) < 0 \Rightarrow$ flow is right to left
- $f(x, t) > 0$ is a source at x , quantity created at x

- $f(x,t) < 0$ is a sink at x ,
quantity destroyed at x . (10)

- Can get a PDE for u by eg:
 - Specifying $f(x,t)$
 - Choosing a relationship between ϕ and u , u_x , etc.
(called a constitutive law)

e.g: $\phi = cu$ gives advection

Diffusion (Logan 1.3)

- Conservation law with no sources:

$$u_t + \phi_x = 0$$

- Suppose $u(x,t)$ is the density of a gas of particles. Then:
 - particles move from high u to low u
 - flux increases with density gradient

→ leads to Fick's law:

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$$\phi(n,t) = -D u_n$$

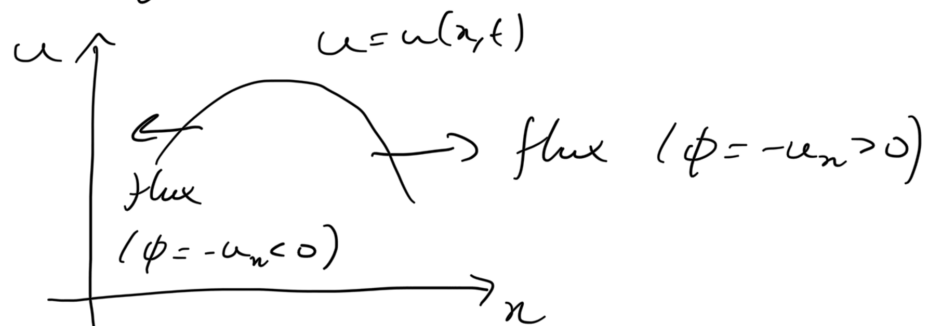
where $D > 0$ is diffusion const.

This gives the diffusion eq.:

$$u_t = D u_{nn}$$

• Same as heat eq.

Intuitive picture:



If $u_{nn} < 0$ (concave down)

then $u_t < 0$ (concentration decreases)

Remark:

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- Can combine diffusion with other effects:

e.g. advection-diffusion:

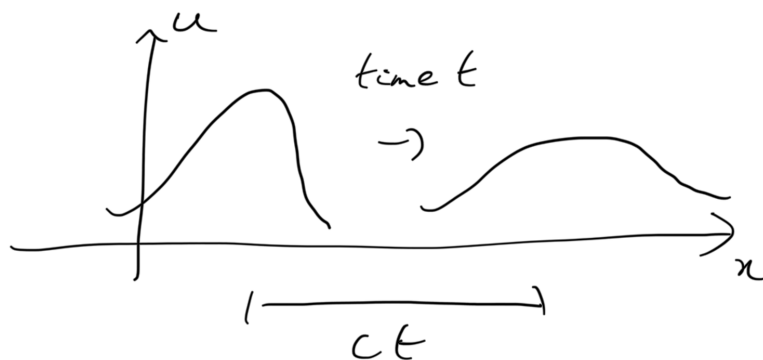
$$\text{flux } \phi = cu - Du_x$$

 ↑ ↑
 advection diffusion

Balance law:

$$u_t + \phi_x = 0$$

$$\Rightarrow u_t + cu_x = Du_{xx}$$



• Rmk:

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diffusion eqs typically come with
initial conditions (ICs) and
boundary conditions (BCs).