

Lecture 2

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Recap:

- PDEs: diff eqs for functions with more than 1 indep. var
- e.g. Heat eq. for $u(x,t)$:

$$u_t - k u_{xx} = 0, \text{ const. } k > 0$$

write as: $Lu = 0$

where $L = (\partial_t - k \partial_x^2)$ is
a differential operator

- L linear $\Leftrightarrow L(u+v) = Lu + Lv$
 $L(cu) = cLu$

$\forall$ const c , functions u, v

- PDE: $Lu = f$ is linear if L is linear
- Linear PDE: $Lu = f$ is homog. if $f = 0$
inhomog. otherwise

First order linear PDE

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- Consider functions $u(x, y)$

$$u: \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$(x, y) \mapsto u(x, y)$$

[Often think of x, y as 'space' variables
and t as 'time' variable]

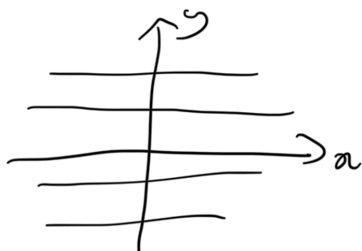
The 'simplest' PDE is:

$$u_x = 0 \quad (\text{1st order, linear, homog.})$$

No explicit y -dependence $\rightarrow$ looks like ODE.

Can solve with direct integration:

$$u(x, y) = f(y) \quad (\text{arbitrary function of } y)$$



$\text{sol}^n u$ is const. on
horizontal lines ($y = \text{const.}$)

Now consider:

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$$au_x + bu_y = 0 \quad (\text{linear, 1st order homog.})$$

for const. $a, b \in \mathbb{R}$

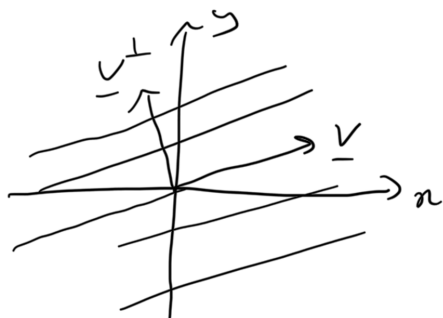
Let $\underline{V} = (a, b) \in \mathbb{R}^2$ (recall $\nabla u = (u_x, u_y)$)

$$\begin{aligned} \text{Then } \underline{V} \cdot \nabla u &= (a, b) \cdot (u_x, u_y) \\ &= au_x + bu_y \\ &= 0 \end{aligned}$$

$\Rightarrow$ Directional derivative in direction

$\underline{V}$ is 0

$\Rightarrow u$ is const. on lines parallel to $\underline{V}$



$$\text{Let } \underline{V}^\perp = (b, -a)$$

$$\text{so } \underline{V} \cdot \underline{V}^\perp = ab - ab = 0$$

Lines parallel to $\underline{V}$ satisfy

$$\underline{V}^\perp \cdot (x, y) = bx - ay = \text{const.}$$

Then u const. on lines parallel to $\underline{v}$ ④

$$\Rightarrow u(x, y) = f(bx - ay)$$

for arbitrary function $f: \mathbb{R} \rightarrow \mathbb{R}$

Check:

$$(a\partial_x + b\partial_y) f(bx - ay)$$

$$= abf' - ba f' = 0$$

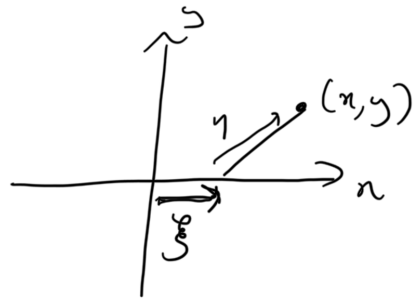
[f' is the derivative of f]

Characteristic Co-ordinates

In this example, the lines parallel to $\underline{v}$ are called characteristics. Here u is constant along characteristics.



We can write points (x, y) in ⑤
characteristic co-ords, (ξ, η) where:



• η is distance along characteristic

• ξ labels characteristic by x -intercept

$$\text{So: } x = \xi + a\eta = x(\xi, \eta)$$

$$y = b\eta = y(\xi, \eta)$$

Transform PDE to (ξ, η) co-ords, $u = u(\xi, \eta)$:

$$\text{Chain rule: } \frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \eta}$$

$$\Rightarrow u_\eta = a u_x + b u_y = 0$$

Using characteristic co-ords, (ξ, η) , we have transformed to the PDE:

$$u_\eta = 0, \text{ for } u(\xi, \eta)$$

$$\text{Sol}^n: u = f(\xi)$$

$$\text{where } \xi = \frac{1}{b}(bx - ay)$$

as before

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Remarks

- η is a co-ordinate along the characteristics. u is const. along characteristics so $u_\eta = 0$
- ξ labels the characteristics. There are other choices for ξ but any choice $\tilde{\xi}$ would have $\tilde{\xi} = g(\xi)$

Method of Characteristics

Method for solving PDEs of form:

$$a(x, y) u_x + b(x, y) u_y = 0$$

Idea: let $\underline{A}(x,y) = (a(x,y), b(x,y))$ ⑦

The directional deriv. $\underline{A} \cdot \nabla u = 0$

$\Rightarrow u$ is constant along curves with slope $\underline{A}$. (characteristics)

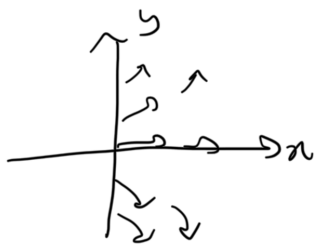
$\rightarrow$ Find co-ord η along characteristics
co-ord ξ labeling //

$\rightarrow$ PDE transforms to $u_\eta = 0$, so $u = f(\xi)$

Eg: $u_x + y u_y = 0$

The characteristic curves $(x(\eta), y(\eta))$

satisfy: $\frac{dx}{d\eta} = 1, \frac{dy}{d\eta} = y$



At $\eta = 0$, take $(x,y) = (0, \xi)$
(ie label characteristics by y -intercept)

$$\Rightarrow \frac{dx}{d\eta} = 1, \quad \frac{dy}{d\eta} = y$$

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$$x(0) = 0, \quad y(0) = \xi$$

$$\Rightarrow x = \eta, \quad y = \xi e^{\eta}, \quad \xi = y e^{-x}$$

$$\text{Then } u_{\eta} = u_x + \xi e^{\eta} u_y$$

$$= u_x + y u_y = 0$$

$$\Rightarrow u = f(\xi) = f(y e^{-x})$$

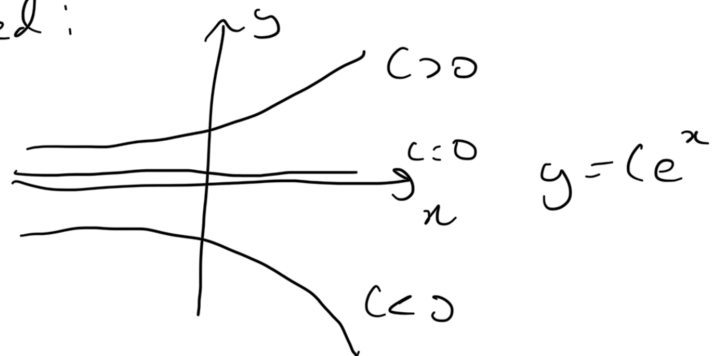
Alternatively:

$$\text{from parametric eqs: } \frac{dx}{d\eta} = 1, \quad \frac{dy}{d\eta} = y$$

$$\text{we find: } \frac{dy}{dx} = y$$

$$\Rightarrow y = C e^x, \text{ const. } C$$

This family covers whole plane when ⑨
 C is varied:



u const. along characteristics:

$$\Rightarrow u(x, Ce^x) = u(0, C)$$

$$= u(0, e^{-x}y)$$

$$= f(e^{-x}y), \text{ arbitrary } f$$

Eg: $u_x + 2xy^2 u_y = 0$

Characteristic curves $(x(\eta), y(\eta))$

satisfy: $\frac{dx}{d\eta} = 1, \quad \frac{dy}{d\eta} = 2xy^2$

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$$\Rightarrow \frac{dy}{dx} = 2xy^2$$

$$\Rightarrow \int \frac{1}{y^2} dy = \int 2x dx$$

$$= -\frac{1}{y} = x^2 - C$$

$$\Rightarrow y = \frac{1}{C - x^2}$$

u const. along characteristics

$$\Rightarrow u(x, y) = F(C) = f\left(x^2 + \frac{1}{y}\right)$$

arbitrary f

Advection (transport)

Consider function $u(x, t)$ describing a quantity u at position x and time t

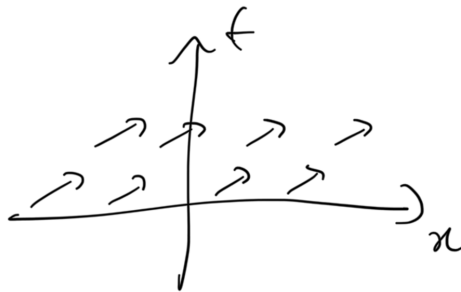
Advection eq:

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$$u_t + cu_x = 0, \quad \text{const. } c > 0$$

Characteristics $(t(\tau), x(\tau))$ satisfy:

$$\frac{dt}{d\tau} = 1, \quad \frac{dx}{d\tau} = c$$



At $\tau = 0$, set

$$(x, t) = (\xi, 0)$$

i.e. label characteristics by x -intercept.

Then:

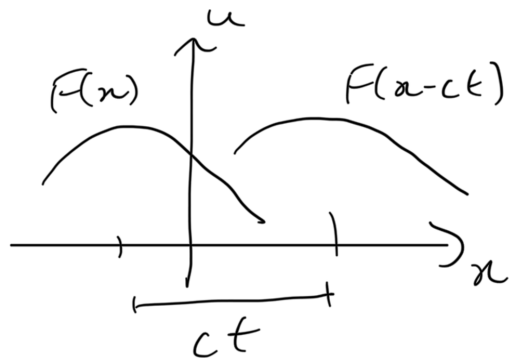
$$t = \tau, \quad x = c\tau + \xi$$

$$\text{and } u_\tau = u_t + u_x \cdot c = 0$$

$$\Rightarrow u = f(\xi) = f(x - ct)$$

Suppose $u(x, 0) = F(x)$ (IC) (12)

Then $u(x, t) = F(x - ct)$



Solution looks like a wave moving right at speed c .

Remark:

- The eq: $u_t + c(x, t)u_x$ describes advection with a spatially varying speed $c(x, t)$.

Ex: Advection & decay

$$u_t + cu_x = -\lambda u$$

↑
wave moves
at speed c

↑
decay at
rate λ

Use characteristic co-ords to simplify (13)
PDE. As before, set:

$$t = \tau, \quad x = c\tau + \xi$$

Then: $u_\tau = u_t + cu_x = -\lambda u$

$$\Rightarrow u_\tau = -\lambda u \quad (\text{PDE for } u(\xi, \tau))$$

No explicit ξ -dependence $\rightarrow$ solve like ODE

$$\begin{aligned} u &= f(\xi) e^{-\lambda \tau} \\ &= f(x - ct) e^{-\lambda t} \end{aligned}$$

$\uparrow$ move right $\nwarrow$ decay

Remark:

• In general, the PDE: $u_t + cu_x = f(t, x, u)$

can be transformed into an ODE using

$$u_\tau = u_t + cu_x \Rightarrow u_\tau = \tilde{f}(\tau, \xi, u)$$

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