

Lecture 11

①

Heat eq. Cauchy problem:

$$u_t - k u_{xx} = 0, \quad k > 0, \quad x \in \mathbb{R}, \quad t > 0$$

$$I(: \quad u(x, 0) = \varphi(x)$$

Wave eq. Cauchy problem:

$$u_{tt} - c^2 u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0$$

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x)$$

What about Laplace's eq?

$$u_{xx} + u_{yy} = 0, \quad u = u(x, y)$$

Let's try and formulate a Cauchy problem with y acting as time:

$$u_{xx} + u_{yy} = 0, \quad x \in \mathbb{R}, \quad y > 0$$

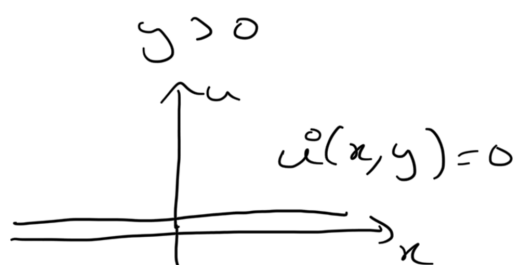
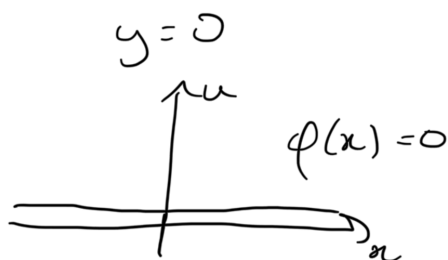
$$\text{with } \underbrace{u(x, 0) = \varphi(x)}_{\text{'initial profile'}} \quad \underbrace{u_y(x, 0) = \psi(x)}_{\text{'initial vel.'}}$$

This is fine for zero-ICs:

(2)

$$\text{Take } \psi(x) = 0, \quad \varphi(x) = \varphi^0(x) = 0$$

$$\text{Then } u(x, y) = u^0(x, y) = 0 \quad \forall y.$$



[Note superscripts are label, not exponent]

• Zero-ICs give zero solⁿ

Now try 'almost zero' ICs:

$$\varphi(x) = \varphi^\lambda(x) = \lambda \cos\left(\frac{x}{\lambda}\right), \quad \psi(x) = 0$$



Let $u^\lambda(x, y)$ be the solⁿ.

$$\text{We have: } \lim_{\lambda \rightarrow 0} \varphi^\lambda(x) = \varphi^0(x) = 0$$

does $\lim_{\lambda \rightarrow 0} u^\lambda(x, y) \stackrel{?}{=} 0$? No! ③

Why:

We can check that (Ex)

$$u^\lambda(x, y) = \lambda \cos\left(\frac{x}{\lambda}\right) \cosh\left(\frac{y}{\lambda}\right)$$

This solⁿ blows up as $\lambda \rightarrow 0$!

$$\text{e.g.: } \lim_{\lambda \rightarrow 0} u^\lambda(0, y) = \lim_{\lambda \rightarrow 0} \frac{\cosh\left(\frac{y}{\lambda}\right)}{1/\lambda}$$

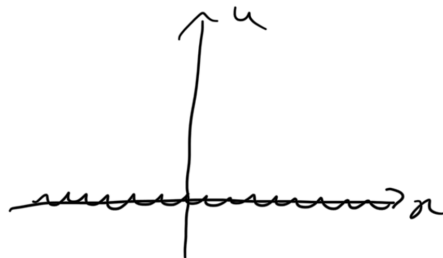
$$= \lim_{\lambda \rightarrow 0} \frac{-\frac{1}{\lambda^2} \sinh\left(\frac{y}{\lambda}\right)}{-1/\lambda^2} \quad (\text{L'Hopital})$$

$$= \lim_{\lambda \rightarrow 0} \sinh\left(\frac{y}{\lambda}\right) \rightarrow \infty$$



$$\varphi^0(x) = 0, \quad \psi(x) = 0$$

$$u^0(x, y) = 0$$



$$\varphi^\lambda(x) = \lambda \cos\left(\frac{x}{\lambda}\right), \quad \psi(x) = 0$$

$$u^\lambda(x, y) = \lambda \cos\left(\frac{x}{\lambda}\right) \cosh\left(\frac{y}{\lambda}\right)$$

$$u^\lambda(0, y) \rightarrow \infty \text{ as } \lambda \rightarrow 0$$

- ④
- The Cauchy problem (IVP for Laplace's eq. is ill-posed!
 - Elliptic PDEs like Laplace's eq. describe static, time-independent processes
 - A PDE with ICs or BCs is well-posed if:
 - 1) Existence (sol^n exists)
 - 2) Uniqueness (sol^n is unique)
 - 3) Stability (sol^n depends continuously on ICs and BCs)

The Cauchy problems for heat, wave eqs are well-posed.

Next: The sol^n s to the wave & heat eq. let us solve more general problems:

- Semi-infinite domains , • Inhomog. problems

Semi-infinite domains (Logan 2.4) (5)

We've solved the heat/wave eqs on the domain $x \in \mathbb{R}$. We can extend these approaches to solve PDEs on the half-line, $x \in (0, \infty)$.

Ex: (*)

$$\text{PDE: } u_t - k u_{xx} = 0 \quad (\text{heat eq.})$$

$$\text{Domain (half line): } x > 0, t > 0$$

$$\text{IC: } u(x, 0) = \phi(x) = x e^{-x^2}, \quad x > 0$$

$$\text{BC: } u(0, t) = 0, \quad t > 0 \quad (\text{Dirichlet})$$

Note: The half-line $(0, \infty)$ has a bdry at $x=0$, so we have a BC for $x=0$



Method: Extend to an IVP on $\mathbb{R}$ ⑥

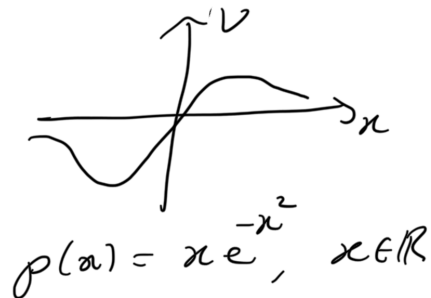
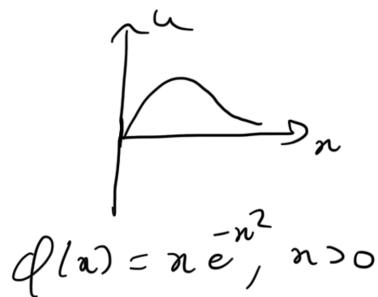
using symmetry, whose solⁿ on $x > 0$ is the solⁿ to (*).

Consider the heat eq. on $\mathbb{R}$:

$$v_t - kv_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0$$

$$\text{with } v(x, 0) = p(x) = \begin{cases} \varphi(x), & x > 0 \\ -\varphi(-x), & x < 0 \end{cases}$$

p is called the odd extension of φ



The solⁿ is:

$$v(x, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{4kt}} p(x - s\sqrt{4kt}) ds$$

The solⁿ $v(x, t)$ satisfies the half-line PDE when restricted to $x > 0$.

Check: $V_t - k V_{xx} = 0$ for $x > 0$

(7)

$$V(x, 0) = \rho(x) = \phi(x) \text{ for } x > 0$$

$$V(0, t) = 0 \quad \forall t \text{ 'by symmetry'}$$

$\Rightarrow$ The solⁿ to (*) is:

$$u(x, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-s^2} \rho(x - s\sqrt{4kt}) ds, \text{ for } x > 0$$

Remark: This works even if $\phi(0) \neq 0$.

Odd & even functions

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function

f is an even function if $f(x) = f(-x)$

|| odd function if $f(x) = -f(-x)$

e.g. even: $1, x^2, \cos x$. odd: $x, x^3, \sinh(x)$

• If f odd and cts, then $f(0) = 0$

• If f even and smooth, then $f'(0) = 0$

Defⁿ: Let $g: [0, \infty] \rightarrow \mathbb{R}$ be a function. (8)

• The odd extension of g is the

function $G_{\text{odd}}(x) = \begin{cases} g(x), & x > 0 \\ -g(-x), & x < 0 \end{cases}$

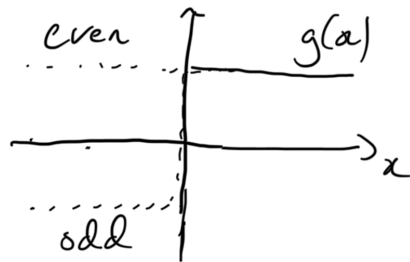
• The even extension of g is the

function $G_{\text{even}}(x) = \begin{cases} g(x), & x > 0 \\ g(-x), & x < 0 \end{cases}$

E.g: $g(x) = 1$, for $x > 0$

Even extension: $G_{\text{even}}(x) = 1$

odd || : $G_{\text{odd}} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$



Symmetry of PDEs

Suppose $u(x, t)$ is a solⁿ of the wave eq.
or heat eq. for $x \in \mathbb{R}$, $t > 0$.

If the initial data are odd ⑨
 functions then $u(x, t)$ is odd wrt. x
 $\Rightarrow u(x, t) = -u(-x, t)$
 $\Rightarrow u(0, t) = 0$ (if u cts at 0)

Similarly:
even initial data $\Rightarrow u(x, t)$ is even wrt x
 $\Rightarrow u_x(0, t) = 0 \quad \forall t.$
 (if u smooth at 0)

Solⁿ of half-line problems ($x \in (0, \infty)$)

Idea: extend PDE to $\mathbb{R}$ by taking
 an odd or even extension of the initial
 data, depending on BCs.

Dirichlet BC $\rightarrow$ odd extension $\rightarrow$ solve, take
 $u(0, t) = 0$ of ICs $x > 0$

Neumann BC $\rightarrow$ even extension $\rightarrow$ solve, take $x > 0$ (10)
 $u_x(0, t) = 0$ of ICs

Eg: Wave eq.

$$u_{tt} - c^2 u_{xx} = 0, \quad x > 0, \quad t > 0$$

$$IC: \quad u(x, 0) = \phi(x) = \cos x, \quad \psi(x) = 0$$

$$BC: \quad u_x(0, t) = 0 \quad (\text{Neumann})$$

Solⁿ: Neumann $\rightarrow$ even extension

Let $p(x), q(x)$ be the even extensions
of $\phi(x), \psi(x)$.

$$\text{Then } p(x) = \begin{cases} \phi(x), & x > 0 \\ \phi(-x), & x < 0 \end{cases} \Rightarrow p(x) = \cos(x)$$

and $\psi(x) = 0$

Solve PDE on $\mathbb{R}$:

$$v_{tt} - c^2 v_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0$$

$$I(1: \quad v(x, 0) = \cos x, \quad v_t(x, 0) = 0 \quad \textcircled{11}$$

D'Alembert:

$$\begin{aligned} v(x, t) &= \frac{1}{2} (p(x-ct) + p(x+ct)) \\ &= \frac{1}{2} (\cos(x-ct) + \cos(x+ct)) \end{aligned}$$

$$= \cos(x) \cos(ct)$$

$$\begin{aligned} \Rightarrow u(x, t) &= v(x, t) \text{ for } x > 0 \\ &= \cos(x) \cos(ct) \end{aligned}$$

$$\begin{aligned} \text{BC is satisfied: } u_x &= -\sin x \cdot \cos(ct) \\ &= 0 \text{ for } x = 0 \end{aligned}$$

Inhomogeneous PDEs

We can use the solⁿs to heat & wave eqs to solve inhomog. PDEs.

Consider the inhomog. heat eq. (12)

with source $f(x, t)$:

$$\begin{aligned} u_t - ku_{xx} &= f(x, t), \quad k > 0 \\ x \in \mathbb{R}, \quad t > 0 \\ u(x, 0) &= 0 \end{aligned} \quad (+)$$

- The solⁿ at time t is the response to a source $f(x, \tau)$ at all $0 < \tau < t$.
- We could therefore get $u(x, t)$ by superposing solⁿs to the heat eq. with IC $f(x, \tau)$ for $0 < \tau < t$.
- Let $w(x, t; \tau)$ be this solⁿ, so:

$$\begin{aligned} w_t - kw_{xx} &= 0, \quad x \in \mathbb{R}, \quad t > 0 \\ w(x, 0; \tau) &= f(x, \tau), \quad x \in \mathbb{R} \end{aligned}$$

The value $w(x, t - \tau; \tau)$ gives the effect of source $f(x, \tau)$ at time $t > \tau$.

Then the solⁿ to (†) is:

(13)

$$u(x, t) = \int_0^t w(x, t-\tau; \tau) d\tau$$

This is Duhamel's principle

We can write $w(x, t; \tau)$ in terms of the heat kernel:

$$G(x, t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}$$

$$\Rightarrow w(x, t; \tau) = \int_{-\infty}^{\infty} G(x-y, t) f(y, \tau) dy$$

$$\Rightarrow u(x, t) = \int_0^t d\tau \int_{-\infty}^{\infty} G(x-y, t-\tau) f(y, \tau) dy$$