

Lecture 10

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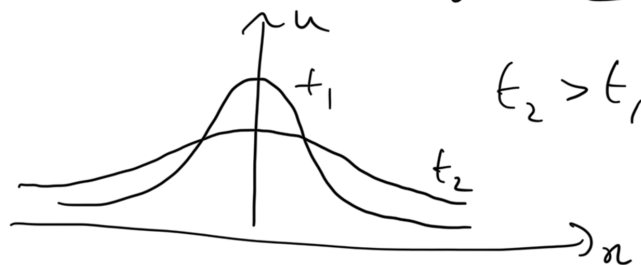
Recap: Heat eq. on $\mathbb{R}$:

$$u_t - k u_{xx} = 0, \quad k > 0, \quad x \in \mathbb{R}, \quad t > 0$$

$$u(x, 0) = \phi(x) \text{ bounded}$$

$$\text{Sol}^n: \quad u(x, t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}} \phi(y) dy$$

* Sol^n describes $u(x, t)$ 'spreading out'



If $u_x \rightarrow 0$ as $|x| \rightarrow \infty$ then the 'total mass' is conserved:

$$M(t) = \int_{-\infty}^{\infty} u(x, t) dx \text{ is conserved}$$

Cauchy problem for wave eq. (Logan 2.2) (2)

The wave eq. for $u(x, t)$ is:

$$u_{tt} - c^2 u_{xx} = 0, \text{ const. } c > 0$$

• 1 space-dim $\rightarrow$ called 1d wave eq.

We can find the general solⁿ more directly than the diffusion case:

Method 1: Let $\xi = x - ct$, $\eta = x + ct$

(Same transformation as HW 3)

HW 3: Write $u(x(\xi, \eta), t(\xi, \eta))$, find $u_{\xi\xi}$...

Here: Write $u(\xi(x, t), \eta(x, t))$:

$$\Rightarrow u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta$$

$$\begin{aligned} u_{xx} &= \partial_\xi (u_\xi + u_\eta) \cdot \xi_x + \partial_\eta (u_\xi + u_\eta) \cdot \eta_x \\ &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \end{aligned}$$

$$\text{and } u_t = -cu_\xi + cu_\eta$$

$$u_{tt} = c^2 u_{\xi\xi} - 2c^2 u_{\xi\eta} + c^2 u_{\eta\eta} \quad (3)$$

$$\Rightarrow u_{tt} - c^2 u_{\eta\eta} = -4c^2 u_{\xi\eta} = 0$$

$$\text{So } u_{tt} - c^2 u_{\eta\eta} = 0 \Leftrightarrow u_{\xi\eta} = 0$$

$$\text{Solve } u_{\xi\eta} = 0$$

$$\Rightarrow u_{\xi} = f(\xi) \quad (\int \text{ wrt } \eta)$$

for arbitrary function f

$$\Rightarrow u(\xi, \eta) = \int f(\xi) d\xi + G(\eta)$$

for arbitrary function G

Integral of arbitrary function is an arbitrary function:

$$\Rightarrow u(\xi, \eta) = F(\xi) + G(\eta)$$

$$\Rightarrow u(x, t) = F(x-ct) + G(x+ct) \quad (4)$$

for arbitrary functions F, G .

Method 2:

$$\text{Let } L = \partial_t^2 - c^2 \partial_x^2 \quad (\text{operator notation})$$

$$\text{So } Lu = (\partial_t^2 - c^2 \partial_x^2)u = u_{tt} - c^2 u_{xx}$$

~~~~~  
factorize?

$$\text{Let } L_1 = \partial_t - c\partial_x, \quad L_2 = \partial_t + c\partial_x$$

$$\text{Lemma: } L = L_1 L_2$$

Proof:  $L_1 L_2 u$  means: apply  $L_2$ , then  
apply  $L_1$  (Like matrices)

$$\text{So: } L_1 L_2 u = L_1 [(\partial_t + c\partial_x)u]$$

$$= L_1 [u_t + cu_x]$$

$$= (\partial_t - c\partial_x)[u_t + cu_x]$$

$$\begin{aligned}
 &= \partial_t [u_t + cu_n] - c \partial_n [u_t + cu_n] \quad (5) \\
 &= u_{tt} - c^2 u_{nn} = Lu
 \end{aligned}$$

Then  $Lu = L_1 L_2 u \quad \forall u(x, t)$

$$\Leftrightarrow L = L_1 L_2$$

How to solve  $Lu = 0 \Leftrightarrow L_1 L_2 u = 0$ ?

1) Let  $v = L_2 u$ , solve  $L_1 v = 0$   
to find  $v$ .

2) Solve  $L_2 u = v$

D'Alembert's sol<sup>n</sup> of the wave eq:

The Cauchy problem for the wave eq.  
is:  $u_{tt} - c^2 u_{xx} = 0, \quad x \in \mathbb{R}, t > 0$

ICs:  $u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x)$

e.g:  $u(x, t)$  is the height of an

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infinite string with initial displacement  $\phi(x)$  and initial vel.  $\psi(x)$ .

Sol<sup>n</sup>:

The general sol<sup>n</sup> to the wave eq. is:  $u(x,t) = F(x-ct) + G(x+ct)$

Use ICs:

$$u(x,0) = F(x) + G(x) = \phi(x)$$

$$u_t(x,0) = -cF'(x) + cG'(x) = \psi(x)$$

These can be solved for  $F, G$ , giving D'Alembert's sol<sup>n</sup>:

$$u(x,t) = \frac{1}{2} \left( \phi(x-ct) + \phi(x+ct) \right) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds$$

Eg: Plucking an infinite string:

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Solve  $u_{tt} - 4u_{xx} = 0$ ,  $x \in \mathbb{R}$ ,  $t > 0$

with  $u(x, 0) = \varphi(x) = e^{-x^2}$

$u_t(x, 0) = \psi(x) = 0$



zero initial vel.

D'Alembert's sol<sup>n</sup>:

$$u(x, t) = \frac{1}{2} \left( \varphi(x-ct) + \varphi(x+ct) \right) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds$$

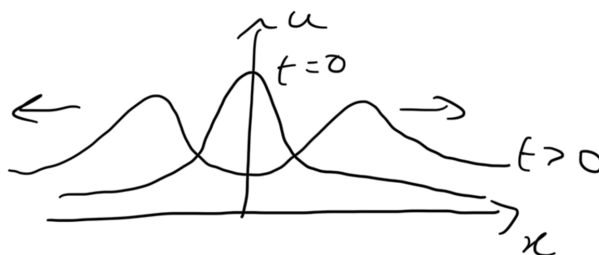
$$= \frac{1}{2} e^{-(x-2t)^2} + \frac{1}{2} e^{-(x+2t)^2}$$

peak at  $x=2t$

→ right moving

peak at  $x=-2t$

→ left moving



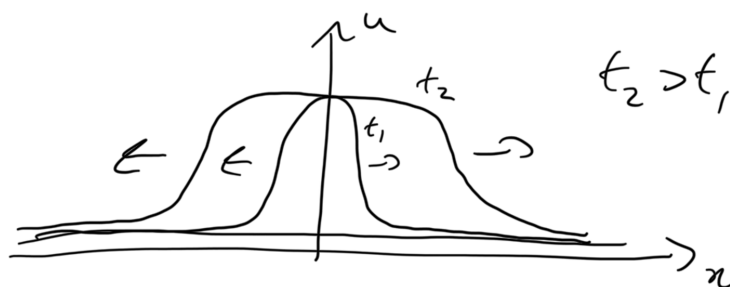
Eg:  $u_{tt} - c^2 u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0$

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$$u(x, 0) = \varphi(x) = 0, \quad u_t(x, 0) = \psi(x) = \frac{1}{1+x^2}$$

IC:  initial vel:  $\psi(x)$

$$\Rightarrow u(x, t) = \int_{x-ct}^{x+ct} \frac{1}{1+s^2} ds = \arctan(x+ct) - \arctan(x-ct)$$



## Causality

D'Alembert's sol<sup>n</sup> reveals the causal structure of the wave eq. (take  $c > 0$ )

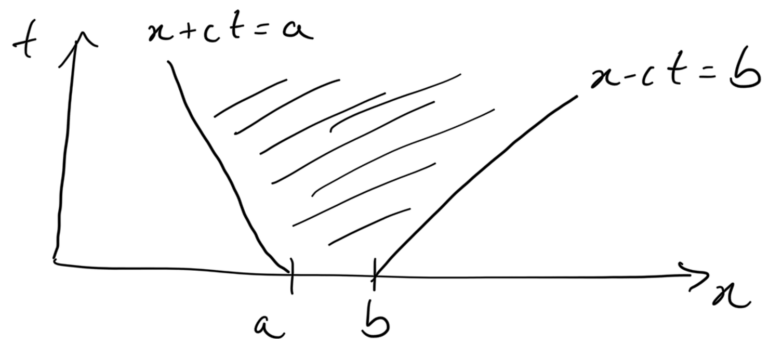
$$u(x, t) = \frac{1}{2} \left( \varphi(x-ct) + \varphi(x+ct) \right) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds$$

for  $u(x,0) = \phi(x)$ ,  $u_t(x,0) = \psi(x)$  (9)

Suppose  $\phi(x), \psi(x) = 0$  for  $x < a$  or  $x > b$   
i.e.  $\phi, \psi$  non-zero only when  $x \in [a, b]$

Then  $u(x,t)$  is non-zero only if

$$x - ct \in [a, b] \text{ or } x + ct \in [a, b]$$



- Signals propagate at speed  $\pm c$ ,  
i.e. along lines:

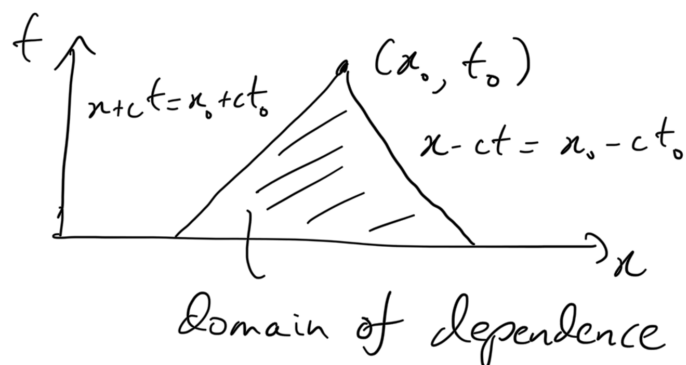
$x - ct = \text{const.}$  (positive characteristics)

$x + ct = \text{const.}$  (negative " )

- Shaded area is region of influence
- The sol<sup>n</sup> at a point  $(x_0, t_0)$  has

a domain of dependence which comes from tracing chars backwards:

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Energy:

The wave eq. has a conserved energy:

Suppose  $u_{tt} - c^2 u_{xx} = 0$ , with

$u_t$  or  $u_x \rightarrow 0$  as  $|x| \rightarrow \infty$

Define: 
$$E(t) = \int_{-\infty}^{\infty} \left( \frac{1}{2} u_t^2 + \frac{c^2}{2} u_x^2 \right) dx$$

Then: 
$$\frac{\partial E}{\partial t} = \int_{-\infty}^{\infty} (u_t u_{tt} + c^2 u_x u_{tx}) dx$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} (c^2 u_t u_{nn} + c^2 u_n u_{tn}) dx \quad (11) \\
&= c^2 \int_{-\infty}^{\infty} \partial_n (u_t u_n) dx \\
&= c^2 [u_t u_n]_{-\infty}^{\infty} = 0
\end{aligned}$$

- Conservation of  $E(t)$  is a good check on a sol<sup>n</sup> (especially for simulations)
- Conserved quantities tell us how a sol<sup>n</sup> behaves, e.g.  $u_n$  can't get 'too large'

### Summary:

- Heat eq.  $\rightarrow$  parabolic, signal spreads out, total mass conserved
- Wave eq.  $\rightarrow$  hyperbolic, coherent signals propagate at speed  $\pm c$ , energy conserved.

What about Laplace's eq.?

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$$u_{xx} + u_{yy} = 0, \quad u = u(x, y)$$

Let's try to formulate a Cauchy problem as with the other eqs, with  $y$  acting as time.

The initial value problem is:

$$u_{xx} + u_{yy} = 0, \quad x \in \mathbb{R}, \quad y > 0$$

with  $\underbrace{u(x, 0) = \varphi(x)}_{\text{initial profile}}, \quad \underbrace{u_y(x, 0) = \psi(x)}_{\text{initial 'velocity'}}$

Eg:  $\varphi(x) = \varphi'(x) = 0$   
 $\psi(x) = 0$



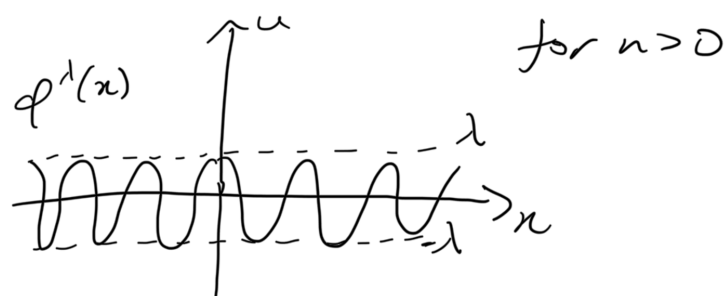
$$\Rightarrow u(x, y) = 0 \quad (\text{superscripts are labels})$$

Zero ICs give zero sol<sup>n</sup>.

Now try 'almost zero' ICs:

$$\varphi(x) = \varphi^\lambda(x) = \lambda \cos \frac{1}{\lambda} x, \quad \psi(x) = 0$$

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Let the sol<sup>n</sup> be  $u^\lambda(x, y)$

We have  $\lim_{\lambda \rightarrow 0} \varphi^\lambda \rightarrow \varphi^0 = 0$

does  $\lim_{\lambda \rightarrow 0} u^\lambda \rightarrow u^0 = 0$ ?

In fact,  $u^\lambda(x, y) = \lambda \cos\left(\frac{x}{\lambda}\right) \cosh\left(\frac{y}{\lambda}\right)$ . (Ex)

This sol<sup>n</sup> blows up as  $\lambda \rightarrow 0$ !

Why?

$$\lim_{\lambda \rightarrow 0} u^\lambda(0, y) = \lim_{\lambda \rightarrow 0} \frac{\cosh\left(\frac{y}{\lambda}\right)}{1/\lambda}$$

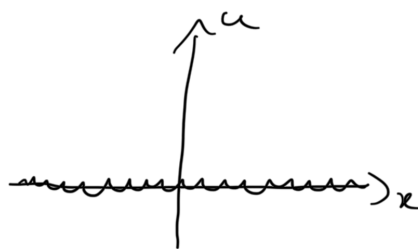
$$= \lim_{\lambda \rightarrow 0} \frac{-\frac{1}{\lambda^2} \sinh\left(\frac{y}{\lambda}\right)}{-1/\lambda^2} \quad (\text{L'Hopital})$$

$$= \lim_{\lambda \rightarrow 0} \sinh\left(\frac{y}{\lambda}\right) \rightarrow \infty$$



$$\phi(x)=0, \psi(x)=0$$

$$u^0(x,y)=0$$



$$\phi^1(x) = \lambda \cos\left(\frac{x}{\lambda}\right), \psi(x)=0$$

$$u^1(x,y) = \lambda \cos\left(\frac{x}{\lambda}\right) \cosh\left(\frac{y}{\lambda}\right)$$

$$u^1(0,y) \rightarrow \infty \text{ as } \lambda \rightarrow \infty$$

The Cauchy problem / initial value problem for Laplace's eq. is ill-posed!

$\Rightarrow$  Laplace's eq. is a static eq. which doesn't describe time dependent processes.