

Lecture 1

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Course organization

- Canvas (see syllabus)
- Regular HWs

Resources

- Discussion sections
- Office hours
- Textbook:
 - * Logan, Applied partial differential eqs.
3rd ed. (2015)

Background

- Logan, appendix A (ODEs)

What is a PDE?

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Recall: ordinary differential equations (ODEs):

- Equations for functions of a single variable $u(t)$, $t \in \mathbb{R}$

e.g. $\frac{du}{dt} = u(1-u)$ is an ODE

$\nearrow$ independent variable $t \in \mathbb{R}$

$\nwarrow$ state / dependent variable u

- In general, an ODE has form

$$F\left(t, u, \frac{du}{dt}\right) = 0$$

- State variable u can have many components

e.g. if $\underline{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \in \mathbb{R}^2$ then

$$\frac{d\underline{u}}{dt} = \underline{u} \text{ is an ODE}$$

PDE: equation for a function with ③
more than 1 independent variable,
e.g. $u(x, y)$

Notation: $u_x = \frac{\partial u}{\partial x}$, $u_y = \frac{\partial u}{\partial y}$, etc.

Examples of PDEs

Heat / diffusion eq. for $u(x, t)$:

$$u_t = k u_{xx} \quad (\text{const. } k)$$

Wave eq. for $u(x, t)$:

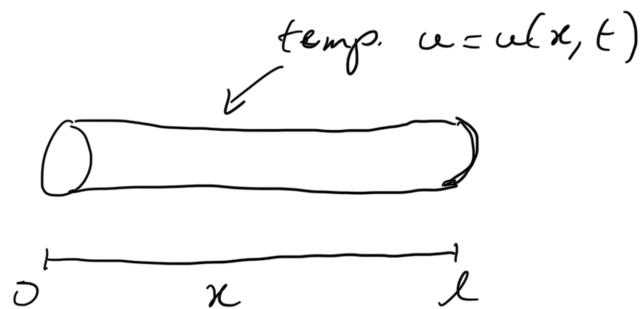
$$u_{tt} = c^2 u_{xx} \quad (\text{const. } c)$$

Laplace's eq. for $u(x, y)$:

$$u_{xx} + u_{yy} = 0$$

- In this class: we will interpret these equations physically, & show how to solve them in different settings (4)

Ex: Heat flow in a bar of length l



- $u(x, t)$ is temp. of bar at position x and time t .
- Suppose the ends are held fixed at temp. 0 and the initial temp. is $\phi(x)$

The temp. evolves according to the heat equation:

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$$u_t = k u_{xx}$$

where $t \geq 0$, $0 \leq x \leq l$ and k is a const. which depends on material.

We also have:

Boundary conditions (BCs):

$$u(0, t) = u(l, t) = 0$$

ends held at temp. 0

Initial condition (IC):

$$u(x, 0) = \phi(x)$$

Rules & defs

- The order of a PDE is the order of the highest derivative that occurs
- Above heat eq. is 2nd order
- A general second order PDE for $u(x, t)$ is:
$$b(x, t, u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}) = 0$$

• A PDE in the above form is:

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linear if G is a linear function
of u and all derivs.

nonlinear otherwise

• A linear PDE is:

homogeneous if every term contains u
or some deriv. of u

nonhomogeneous otherwise

/inhomogeneous

EG: 1) $u_t + e^{x^2 t} u_{xx} = 0$

is linear: dependence on u_t , u_{xx}
is linear

and homogeneous

$$2) \quad u_t + 3xu_{xn} = \frac{t}{x}$$

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is linear and inhomogeneous

3) $u_t + uu_n = 0$ is nonlinear because of the uu_n term

4) $u_{tt} + u_n + \sin^2 u = 0$ is non-linear because of the $\sin^2 u$ term

Operator notation and linearity

We can write heat eq: $u_t - ku_{xx} = 0$

as: $Lu = 0$

where $L = \frac{\partial}{\partial t} - k \frac{\partial^2}{\partial x^2}$

- L is a differential operator ⑧
 L acts on (smooth) functions $u(x, t)$
 to produce a new function

$$L(u) = Lu = \left(\frac{\partial}{\partial t} - k \frac{\partial^2}{\partial x^2} \right) u$$

$$= (\partial_t - k \partial_x^2) u = u_t - k u_{xx}$$

- L is a linear operator if and only if:
 $L(u+v) = Lu + Lv \quad \forall \text{ functions } u, v$

$$L(cu) = cLu \quad \forall \text{ const } c \in \mathbb{R}$$

E.g.:

1) $Lu = u_t + x u_{xx}$ is linear:

$$\text{check: } L(u+v) = \frac{\partial}{\partial t} (u+v) + x \frac{\partial^2}{\partial x^2} (u+v)$$

$$= u_t + v_t + x u_{xx} + x v_{xx}$$

$$= Lu + Lv$$

$$L(u) = cu_t + cu_{xx} = cLu$$

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2) $Lu = u_t + uu_x$ is non-linear:

$$L(cu) = cu_t + cu \cdot cu_x$$

$$= cu_t + c^2 u \cdot u_x$$

$\neq cLu$ in general

Remarks:

• If L is a linear operator then

$Lu = f$ is a linear PDE

e.g: $\underbrace{u_t + uu_{xx}}_{Lu} = \underbrace{e^{x^2 t}}_{f(x,t)}$ is linear

• If L is a linear operator then

the PDE $Lu = f$ is:

homogeneous if $f = 0$

inhomogeneous if $f \neq 0$

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- A linear, homogeneous PDE $Lu=0$ has the superposition principle:

If u_1, u_2 are solutions to the PDE then $au_1 + bu_2$ is a solution for consts a, b

i.e.:

$$Lu_1 = Lu_2 = 0 \Rightarrow L(au_1 + bu_2) = 0$$

- The distinction between linear and nonlinear PDEs is very important.

Solutions to PDEs

- A solution to a PDE is a function

u which satisfies the PDE and ⑪
any boundary / initial conditions
upon direct substitution

- The general solution to an ODE involves arbitrary constants, the general solution to a PDE involves arbitrary functions
- Typically we are interested in specific solutions to PDEs, not general solutions

Eg: 1) Heat eq: $u_t - u_{xx} = 0$

$u_1(x, t) = x^2 + 2t$ is a solution:

$$\partial_t u_1 - \partial_x^2 u_1 = 2 - 2 = 0$$

$u_2(x, t) = e^{-t} \sin x$ is a solution:

$$\partial_t u_2 - \partial_x^2 u_2 = -e^{-t} \sin x + e^{-t} \sin x \quad (12)$$

$$= 0$$

2) $u_x = tx$ (1st order, linear, inhomogeneous)

Can solve by direct integration

• Integrate both sides wrt x :

$$u(x, t) = \frac{1}{2} tx^2 + A(t)$$

$\uparrow$
 arbitrary function
 of t

3) $u_{tt} - 4u = 0$ (2nd order, linear homogeneous)

No explicit x -dependence $\rightarrow$ treat as ODE

with x as a parameter

$\rightarrow$ 'constants' depend on x

General solution:

$$u(x, t) = A(x) e^{-2t} + B(x) e^{2t}$$