

Math 110: Introduction to Partial Differential Equations
Spring quarter 2026

Homework 4

Due: 11:59 pm, Monday, May 11, 2026

Instructions. Please submit your solution as a single PDF file to Gradescope. If you write your solution, please write it neatly, and then scan it into a PDF file.

1. (Gaussian integrals.) The value of $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$. Use this fact to calculate the values of the following integrals (a, b are constants)

(a) $\int_{-\infty}^{\infty} e^{-ax^2} dx$

(b) $\int_{-\infty}^{\infty} e^{-\frac{(x-a)^2}{b^2}} dx$

(c) $\int_{-\infty}^{\infty} e^{-ax^2+2bx} dx$

(d) $\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx$ (Integrate by parts)

2. (Heat equation.) Let $k > 0$ be a constant. Solve the heat equation, $u_t - ku_{xx} = 0$ for $x \in \mathbb{R}$, $t > 0$, for the following initial conditions. Write the solution as a closed form in terms of the error function

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-z^2} dz$$

(a) $u(x, 0) = e^{-\frac{(x-a)^2}{b^2}}$ where a, b are constants.

(b) $u(x, 0) = e^{-|x|}$

(c)

$$u(x, 0) = \begin{cases} 1 - |x| & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

(d)

$$u(x, 0) = \begin{cases} 1 & \text{if } |x-1| < 1/2 \text{ or } |x+1| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

At what time t does $u(0, t)$ attain its maximum value?

3. Suppose $u(x, t)$ satisfies the heat equation, $u_t - ku_{xx} = 0$, for constant $k > 0$, and $x \in \mathbb{R}, t > 0$, with initial condition is $u(x, 0) = \phi(x)$. Suppose $\phi(x) = -\phi(-x)$ for all x (i.e. ϕ is an *odd* function).

(a) Using the integral formula, write down an expression for $u(x, t)$ and $u(-x, t)$.

(b) Show that $u(x, t) = -u(-x, t)$. (Hint: make a substitution in the integral expression for $u(x, t)$.)

(c) Conclude that $u(0, t) = 0$ for all t .

4. (2d heat equation.) Let $f(x, t)$ and $g(x, t)$ be solutions of the heat equation with diffusivity $k > 0$ and initial conditions $\phi(x)$ and $\psi(x)$ respectively:

$$\begin{aligned} f_t - kf_{xx} &= 0 & f(x, 0) &= \phi(x) \\ g_t - kg_{xx} &= 0 & g(x, 0) &= \psi(x) \end{aligned}$$

for all $x \in \mathbb{R}$ and $t > 0$.

- (a) Show that $u(x, t) = f(x, t) + g(x, t)$ is a solution of the heat equation. What is its initial condition?
- (b) Show that $u(x, y, t) = f(x, t)g(y, t)$ is a solution of the 2d heat equation,

$$u_t - k(u_{xx} + u_{yy}) = 0$$

What is its initial condition?

- (c) Hence find a solution of the 2d heat equation for $u(x, y, t)$.

$$\begin{aligned} u_t - ku_{xx} - ku_{yy} &= 0 \\ u(x, y, 0) = \phi(x, y) &= \begin{cases} 1 & \text{if } |x|, |y| < 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

[Hint: find $f(x), g(y)$ such that $\phi(x, y) = f(x)g(y)$.]

5. (Inhomogeneous heat equation.) Let $u(x, t) = 2kt + x^2$, where $k > 0$ is a constant.

- (a) Show that u satisfies the heat equation, i.e. show $(\partial_t - k\partial_x^2)u(x, t) = 0$.
- (b) Find $\phi(x) = u(x, 0)$.
- (c) Find $v(x, t)$, defined by

$$v(x, t) = \int_0^t u(x, t - \tau) d\tau$$

Calculate $(\partial_t - k\partial_x^2)v(x, t)$.

- (d) Hence write down a solution, $w(x, t)$ of the following inhomogeneous heat equation

$$\begin{aligned} w_t - kw_{xx} &= x^2 \\ w(x, 0) &= 0 \end{aligned}$$

for all $x \in \mathbb{R}$ and $t > 0$.