

Math 110: Introduction to Partial Differential Equations
Spring quarter 2026
Homework 3

Due: 4:00 pm, Friday, May 1, 2026

Instructions. Please submit your solution as a single PDF file to Gradescope. If you write your solution, please write it neatly, and then scan it into a PDF file.

1. (Laplace's equation.)

(a) Let $r(x, y) = \sqrt{x^2 + y^2}$. Find r_x, r_y, r_{xx} and r_{yy} .

(b) Consider Laplace's equation, $\nabla^2 u = 0$, for $u(x, y)$. Suppose u has the form $u(x, y) = f(r(x, y))$ for some function f . Show that

$$\nabla^2 u = f''(r) + \frac{1}{r}f'(r) = 0$$

(c) Find the general solution of this ODE for $f(r)$.

2. (Classification of PDEs.) The function $u(x, t)$ satisfies the wave equation $u_{tt} - c^2 u_{xx} = 0$.

(a) Find the discriminant of this PDE and state whether the PDE is hyperbolic, parabolic or elliptic.

(b) Define ξ, η as follows

$$x = \frac{1}{2}\xi + \frac{1}{2}\eta \quad , \quad t = \frac{1}{2c}\xi - \frac{1}{2c}\eta$$

Find $u_{\xi\xi}, u_{\xi\eta}, u_{\eta\eta}$ (use the fact that u satisfies the wave equation).

(c) Hence find constants A, B, C such that $Au_{\xi\xi} + Bu_{\xi\eta} + Cu_{\eta\eta} = 0$. Is this PDE hyperbolic, parabolic or elliptic?

3. Consider the heat equation, $u_t - ku_{xx} = 0$, for constant $k > 0$.

(a) Let $u(x, t) = 2kt + x^2$. Show that $u(x, t)$ and $u(x - y, t)$ are solutions of the heat equation for any y . Does $u(x, t)$ satisfy the decay condition $u_x \rightarrow 0$ as $x \rightarrow \pm\infty$?

(b) Define $v(x, t)$ as follows

$$v(x, t) = \int_0^1 u(x - y, t) y^2 dy$$

Calculate $v(x, t)$, and show that v solves the heat equation, $v_t - kv_{xx} = 0$.

(c) Show that

$$v_t = \int_0^1 u_t(x - y, t) y^2 dy$$
$$v_{xx} = \int_0^1 u_{xx}(x - y, t) y^2 dy$$

(i.e. integration and differentiation can be exchanged here)

4. Suppose $u(x, t)$ solves the heat equation, $u_t - ku_{xx} = 0$ for constant $k > 0$.
- (a) Let $w(x, t) = u(\sqrt{a}x, at)$ where $a \in \mathbb{R}$ is a constant. Show that w solves the heat equation, $w_t - kw_{xx} = 0$.
 - (b) Suppose that $u(x, t)$ has the form $f(z(x, y))$, where $z(x, y) = x/\sqrt{4kt}$. By substituting $f(z)$ into the heat equation, show that $f(z)$ satisfies an ODE and find this ODE.

5. Consider the function

$$p(x, t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-\mu)^2}{4kt}}$$

where $\mu, k \in \mathbb{R}$ are constants with $k > 0$.

- (a) Show that $p(x, t)$ solves the heat equation, $p_t - kp_{xx} = 0$.
- (b) Show that

$$\int_{-\infty}^{\infty} p(x, t) dx = 1$$

You may use the fact that $\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$ without proof.

($p(x, t)$ is the probability density function of a normally distributed random variable with mean μ and variance $2kt$.)