

Math 110: Introduction to Partial Differential Equations
Spring quarter 2026
Homework 2

Due: 4:00 pm, Friday, April 17, 2026

Instructions. Please submit your solution as a single PDF file to Gradescope. You are encouraged to type out your solution. If you write your solution, please write it neatly, and then scan it into a PDF file.

1. By trial and error, find a solution to the diffusion equation, $u_t = Du_{xx}$, with initial condition $u(x, 0) = x^2$ (where $D > 0$ is a constant).
2. A rod occupying the interval $0 \leq x \leq \ell$ is subject to a heat source $f(x)$. The temperature, $u(x, t)$ of the rod has flux $\phi(x, t) = -Du_x(x, t)$ for constant $D > 0$.

(a) Interpret the terms in the equation for the heat flow $u_t = Du_{xx} + f$

(b) Find the steady-state (time-independent) solution in the following cases:

(i)

$$f(x) = 1 \quad \text{with BCs} \quad u(0, t) = 0, \quad u(\ell, t) = 1$$

(ii)

$$f(x) = \begin{cases} 0 & \text{if } 0 < x < \ell/2 \\ H & \text{if } \ell/2 < x < \ell \end{cases} \quad \text{with BCs} \quad u(0, t) = 0, \quad u(\ell, t) = 0$$

3. Find the general form of the steady-state solution to the following PDEs

(a) $u_t = Du_{xx} - cu_x$

(b) $u_t = Du_{xx} - cu_x + ru$

(D, c, r are positive constants).

4. Consider the advection-diffusion-decay equation, $u_t = Du_{xx} - cu_x - \lambda u$. Define $w(x, t)$ by $u(x, t) = w(x, t)e^{\alpha x - \beta t}$. Show that α and β can be chosen such that w satisfies the diffusion equation. (α and β will depend on D, c, λ .)
5. Let $u(x, t)$ satisfy the diffusion equation $u_t = Du_{xx}$ on $0 < x < \ell$, $t > 0$, and constant $D > 0$. u satisfies the following boundary and initial conditions

$$u(0, t) = u(\ell, t) = 0 \quad t > 0$$

$$u(x, 0) = u_0(x), \quad 0 \leq x \leq \ell$$

Define $E(t)$ as follows

$$E(t) = \int_0^\ell u(x, t)^2 dx$$

Show that $dE/dt \leq 0$. Conclude that

$$\int_0^\ell u(x, t)^2 dx \leq \int_0^\ell u_0(x)^2 dx$$

for $t \geq 0$.