Further results on NC polynomials and heavy convexity

July 2, 2002

Contents

TO DO

Put in example $\{X > 0, Y > 0, XY + YX > 0\}$. DOES NOT HAVE LMI REP.

1 LMI Inequalities - REAL

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SEE PROPERTIES i Poss4.tex

DEF of $\mathcal{C}_{\mathcal{P}}$ in Poss4.tex?? is used only in isolated places.

delete heavy?? thruout since it is implied by results in pos 4.

How is $\mathcal{D}_c P$ bounded used in the proof?

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Define $M^g_{\infty\times\infty}$ to be all g tuples of matrices in $M_{\infty\times\infty}$. We emphasize that any member of such a tuple is a finite dimensional matrix.

If $L \in M_{\infty\times\infty}(\mathcal{N}_*)$ is affine linear and $X = (X_1, \ldots, X_g)$ is a tuple of operators, then $L(X) \geq 0$ is called a **Linear Matrix Inequality (LMI)**. Note if $\mathcal{P}_1$ is a collection of symmetric affine linear polynomials from $M_{l\times l}(\mathcal{N}_*)$, then $\mathcal{D}_{\mathcal{P}_1}$ is an heavy?? convex positivity domain.

Theorem 1.1 *Suppose $\mathcal{P}$ is a collection of polynomials in $M_{\infty\times\infty}(\mathcal{N}_*)$, or in $M_{\infty\times\infty}(\mathcal{N})$, or in $M_{\infty\times\infty}(\mathcal{N}_*\mathcal{N})$. If $\mathcal{D}_{\mathcal{P}}$ is a bounded heavy?? convex positivity domain, then there exists a collection $\mathcal{P}_1$ of symmetric affine linear polynomials from $M_{\infty\times\infty}$??$(\mathcal{N}_*)$ such that*

$$\mathcal{D}_{\mathcal{P}_1} \cap M^g_{\infty\times\infty} = \mathcal{D}_{\mathcal{P}} \cap M^g_{\infty\times\infty}$$

.

Of course a $\mathcal{P}_1$ may be an infinite collection with no bound on the size of the matrix polynomials involved, even if $\mathcal{P}$ is finite. *A conjecture is that $\mathcal{P}_1$ is finite if $\mathcal{P}$ consists of a finite number of polynomials?*

1.1 Arveson gets real

Let H and $\tilde{H}$ be real Hilbert spaces with $\tilde{H}$ a finite dimensional. A real completely positive map ψ on a subspace β of $\mathcal{B}(H)$ into $\mathcal{B}(\tilde{H})$ has a representation

$$\psi(Z) = V^T \pi(Z) V \qquad \forall Z \in \beta$$

with homomorphism π on real spaces $\mathcal{B}(H) \to \mathcal{B}(K)$ and $V : \mathcal{B}(\tilde{H}) \to \mathcal{B}(K)$ an isometry.

SAY MORE: WHAT IS COMPLETELY POS.

Give REFerence OR PROOF FOR REAL CASE that finite dim (or some faximile) is essential to real but not complex case???

1.2 Proof for REAL LMI

First we treat the $M_{\infty\times\infty}(\mathcal{N}_*)$ case. Let $\mathcal{P}_1$ denote the collection of all affine linear (square) matrix-valued $M_{\infty\times\infty}(\mathcal{N}_*)$ symmetric polynomials L such that $L(X) \geq 0$ whenever $X \in \mathcal{D}_{\mathcal{P}}$. The boundedness hypothesis implies that

$$L_j = \begin{pmatrix} C & x_j \\ x_j^* & C \end{pmatrix}$$

is in $\mathcal{P}_1$, since $L_j(X) \geq 0$ if and only if $\|X_j\| \leq C$, it follows that $\mathcal{D}_{\mathcal{P}_1}$ is a bounded positivity domain. It is also heavy?? convex.

Suppose $Y \in \mathcal{D}_{\mathcal{P}}$. If $L \in \mathcal{P}_1$, then $L(Y) \geq 0$ by the definition of $\mathcal{P}_1$ and so $Y \in \mathcal{D}_{\mathcal{P}_1}$. Hence $\mathcal{D}_{\mathcal{P}} \subset \mathcal{D}_{\mathcal{P}_1}$.

To prove the converse (reverse inclusion), fix a separable infinite dimensional Hilbert space $\mathcal{H}$ (so $\mathcal{H}$ is isomorphic to ℓ^2) and let $\mathcal{D}$ denote the tuples $X = (X_1, \ldots, X_g)$ of operators on $\mathcal{H}$ such that $X \in \mathcal{D}_{\mathcal{P}}$. Let $\mathbf{X} = \oplus_{\mathcal{D}} X$ acting on $\mathbf{H} = \oplus_{\mathcal{D}} H$.

SCOTT ?? - could you write a passage justifying the techniclities of this really infinite direct sum.

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UNECESSARY ??? SCOTT

Of course, if $q \in \mathcal{C}_{\mathcal{P}}$, then $q(\mathbf{X}) \geq 0$. Conversely, if $q(\mathbf{X}) > 0$, then $q \in \mathcal{C}_{\mathcal{P}}$ which we prove in the following paragraph.

Suppose $W = (W_1, \ldots, W_g)$ is a tuple of bounded operators acting on the Hilbert space $\mathcal{T}$ and $W \in \mathcal{D}_{\mathcal{P}}$ and assume that $q \in M_{l\times l}(\mathcal{N}_*)$ (so $\ell \times \ell$ matrix valued). Given a non-zero vector $\gamma = \oplus_1^\ell \gamma_j \in \oplus_1^\ell \mathcal{T}$, to see that $< q(W)\gamma, \gamma >$ is positive , let $\mathcal{T}_\gamma$ denote the smallest reducing subspace for $\{W_1, \ldots, W_g, W_1^*, \ldots, W_g^*\}$ containing γ. The subspace $\mathcal{T}_\gamma$ is seperable so there are two cases. First, if $\mathcal{T}_\gamma$ is infinite dimensional, then $\mathcal{T}_\gamma$ is isomorphic to $\mathcal{H}$ and, up to unitary equivalence, we may assume that T_γ, the restriction of W to $\mathcal{T}_\gamma$, is a member of $\mathcal{D}_{\mathcal{P}}$. It follows that, as $q(\mathbf{X}) > 0$, one has $q(W_\gamma) > 0$. Thus $< q(W)\gamma, \gamma > = < q(W_\gamma)\gamma, \gamma >$ is positive. If $\mathcal{T}_\gamma$ is finite dimensional, then replace $\mathcal{T}_\gamma$ with $\oplus_1^\infty \mathcal{T}_\gamma$ and W_γ with $\oplus_1^\infty W_\gamma$ and argue as above. In any event, we have $q(W) > 0$. By the Positivestellensatz, we conclude $q \in \mathcal{C}_{\mathcal{P}}$.

END UNECESSASaRY ??

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Let $\beta(\mathbf{X})$ denote the subspace of $\mathcal{B}(\mathbf{H})$,

$$\beta(\mathbf{X}) = \{l(\mathbf{X}) : l \text{ is a scalar affine linear NC polynomial}\}.$$

Given $Y \in \mathcal{D}_{\mathcal{P}_1}$ acting on the finite dimensional Hilbert space $\mathcal{Y}$, consider the mapping $\psi : \beta(\mathbf{X}) \mapsto \beta(Y)$ defined by

$$\psi(l(\mathbf{X})) = l(Y).$$

To see that this mapping is well defined and completely positive, it is enough to show for any ℓ, if $L \in M_{l\times l}(\beta(\mathbf{X}))$ and if $L(\mathbf{X}) \geq 0$, then $L(Y) \geq 0$.

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THIS RELPLACED WITH ARG BELOW ?? OK??

But, if $L(\mathbf{X}) \geq 0$ and $\epsilon > 0$, then by what was proved above $L + \epsilon \in \mathcal{C}_{\mathcal{P}}$ and thus $L + \epsilon \in \mathcal{P}_1$. We conclude that $L(Y) \geq -\epsilon$ and since $\epsilon > 0$ is arbitrary, $L(Y) \geq 0$.

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NEW

If $L(\mathbf{X}) \geq 0$, then L is nonegative on $\mathcal{D}_{\mathcal{P}}$; so, by the definition of $\mathcal{P}_1$, we have that $L \in \mathcal{P}_1$. Since Y is in $\mathcal{D}_{\mathcal{P}_1}$, we get $L(Y) \geq 0$.

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Since $\beta(\mathbf{X})$ is a symmetric subspace of $\mathcal{B}(\mathbf{H})$ containing the identity and $\psi(I) = I$ (ψ is unital) and completely positive on $\beta(\mathbf{X})$, by the Arveson Extension Theorem and the Steinsprings Representation Theorem, there exists an auxiliary real Hilbert space K, an isometry $V : \mathcal{Y} \mapsto K$, and representation $\pi : \mathcal{B}(\mathbf{H}) \mapsto \mathcal{B}(K)$, such that $\psi(\cdot) = V^*\pi(\cdot)V$. (The fact that this works for H a real space uses that $\mathcal{Y}$ is finite dimensional.) Thus, for affine linear scalar polynomials l, we have $\psi(l(Y)) = V^*\pi(l(\mathbf{X}))V$. In particular, $Y = V^*\pi(\mathbf{X})V$ for an isometry V. (Note that in the construction $\mathbf{X}$ doesn't depend upon Y, but π does.)

If $p \in \mathcal{P}$ and $p = \{q_{\alpha,\beta}\}_{\alpha,\beta}^{\ell} \in M_{l\times l}(\mathcal{N}_*)$, then

$$p(\pi(\mathbf{X})) = \{\pi(p_{\alpha,\beta}(\mathbf{X}))\}_{\alpha,\beta}^{\ell},$$

which is $\pi(p(\mathbf{X})$, or more properly written is $1 \otimes \pi(p(\mathbf{X}))$. Thus $p(\pi(\mathbf{X})) \geq 0$ for all $p \in \mathcal{P}$, and consequently $\pi(\mathbf{X}) \in \mathcal{D}_{\mathcal{P}}$. Thus, by the heavy?? convex hypothesis,

$Y \in \mathcal{D}_{\mathcal{P}}$. Hence $\mathcal{D}_{\mathcal{P}} = \mathcal{D}_{\mathcal{P}_1}$, thereby proving the main result in the Theorem for $M_{\infty\times\infty}(\mathcal{N}_*)$. [1]

To prove the $\mathcal{P} \subset M_{\infty\times\infty}(\mathcal{N})$ case think of $\mathcal{P}$ as a subset of $M_{\infty\times\infty}(\mathcal{N}_*)$ and set $\tilde{\mathcal{P}} := \mathcal{P} \cup \pm(x_j - x_j^T)$. Apply the $M_{\infty\times\infty}(\mathcal{N}_*)$ result to $\tilde{\mathcal{P}}$.

For the hereditary case the proof follows from the $M_{\infty\times\infty}(\mathcal{N}_*)$ case, since affine linear polynomials in $\mathcal{N}_*$ are automatically hereditary. ••

INTRODUCE NEXT corrolary??

Corollary 1.2 *?? BEWARE NEEDS DEF OF $\mathcal{C}_{\mathcal{P}}$??*

Suppose we are in the set up of Theorem 3.4. If $p \in \mathcal{P}$ and $\epsilon > 0$, then $p + \epsilon \in \mathcal{C}_{\mathcal{P}_1}$. Moreover, if $L \in \mathcal{P}_1$ and $\epsilon > 0$, then $L + \epsilon \in \mathcal{C}_{\mathcal{P}}$.

Proof ?? BEWARE NEEDS PSS ?? If $q \in \mathcal{C}_{\mathcal{P}}$ and $\epsilon > 0$, then $q(X) + \epsilon I > 0$ for all $X \in \mathcal{D}_{\mathcal{P}_1}$ and therefore by the noncommutative Positivestelensatz, $q + \epsilon \in \mathcal{C}_{\mathcal{P}_1}$. The proof of the last assertion in the theoerem is almost the same. ••

[1] This result holds more generally than for positivity domains. Namely, for convex domains closed with respect to direct sums and unital representations (including restriction to reducing subspaces.

2 DEFS- Complex

Some additional results on NC polynomials, positivity domains, and LMI inequalities.

Let NCP_* denote the NC polynomials in the non-commutative variables $\{x_1, \dots, x_g, x_1^*, \dots, x_g^*\}$ and let $MNCP_*$ denote the matrix valued NC polynomials in the same variables. To emphasis a particular matrix size, write $M_{m,n}(NCP_*)$ to denote the $m \times n$ matrices with entries from NCP_*.

There is the involution on NCP_* and $p \in MNCP_*$ is self-adjoint if $p^* = p$. Given $\mathcal{P}$, a collection of symmetric polynomials from $MNCP_*$, let $\mathcal{C}_{\mathcal{P}}$ denote the wedge generated by polynomials of the form s^*ps and r^*r where $p \in \mathcal{P}$ and r and s come from $MNCP_*$.

Also, let $\mathcal{D}c_{\mathcal{P}}$ denote the tuples $X = (X_1, \dots, X_g)$, where $X_1, \dots, X_g$ are operators on the same Hilbert space H, such that $p(X) \geq 0$ for all $p \in \mathcal{P}$.

Note $\mathcal{D}c_{\mathcal{P}}$ is really a graded object, graded by the dimension of H, and is called the *positivity domain* of $\mathcal{P}$.

A positivity domain $\mathcal{D}c_{\mathcal{P}}$ is *bounded* if there exists a $C > 0$ such that $\|X_j\| \leq C$ for each $1 \leq j \leq g$ whenever $X = (X_1, \dots, X_g) \in \mathcal{D}c_{\mathcal{P}}$.

END SEMIOLD DEFS

3 Symmetrized hereditary polynomial heavy convex domains

Fix a scalar symmetrized hereditary polynomial of degree d; i.e.,

$$p = I - \sum_{k=1}^{d} \sum_{|w|=k} p_w w^* w, \tag{1}$$

where $|w|$ is the length of the word w in the free semi-group on the symbols $\{x_1, \dots, x_g\}$.

(Note that the positivity domain $\mathcal{D}c_p = \mathcal{D}c_{\{p\}}$ contains the 0 tuple $0 = (0, 0, \dots, 0)$ (for every choice of Hilbert space H) and therefore, if $\mathcal{D}c_p$ is also heavy convex and if $X \in \mathcal{D}c_p$, then $tX \in \mathcal{D}c_p$ for all $|t| \leq 1$. This seems not entirely relevant, though related.)

We will say that $\mathcal{D}c_p$ is **psuedo-bounded** if, for each finite dimensional Hilbert space H the component of the set

$$\{X = (X_1, \dots, X_g) : X_j \in \mathcal{B}(H),\ p(X) > 0\} \subset \mathcal{B}(H)$$

containing 0 is bounded. Here $p(X) > 0$ means $p(X)$ is strictly positive definite. (This definition is somewhat PROVISIONAL ?? Also placed psuedo-bounded in braces {} for easy renaming.) There is of course a similar notion for self-adjoint collections $\mathcal{P}$ instead of just the singleton $\{p\}$.

The positivity domain $\mathcal{D}c_p$ is closed with respect to compression to semi-invariant subspaces if whenever a tuple $X = (X_1, \dots, X_g)$ acting on a Hilbert space H is in $\mathcal{D}c_p$ and P is the projection onto a a subspace which is semi-invariant for each X_j it follows that PXP is in $\mathcal{D}c_p$. Thus, closed with respect to compressions to semi-invariant subspaces is a weaker condition than heavy?? convex.

Theorem 3.1 *(a) If $\mathcal{D}c_p$ is psuedo-bounded and closed with respect to compression to semi-invariant subspaces, then $p_{x_j} > 0$ for each $1 \leq j \leq g$ and $p_w \geq 0$ for all w.*

(b) If $\mathcal{D}c_p$ is psuedo-bounded and heavy convex, then, $p_{x_j} > 0$ for each $1 \leq j \leq g$ and $p_w = 0$ for all $|w| > 1$. In particular, p is quadratic.

Consequently, if p defines a psuedo-bounded heavy convex positivity domain, then $p = I - \sum_{0<|w|\leq d} c_w^2 w^* w$ and, using the Cholesky algorithm, there is a $(D+1) \times (D+1)$ matrix-valued linear polynomial L such that p and L determine the same positivity domain: $\mathcal{D}c_p = \mathcal{D}c_L$. To describe L, let $D = \sum_{n=1}^{d} g^n$ and view $\mathbb{C}^D$ as the Hilbert space with orthonormal basis $\{w : w \text{ is a word }, 0 < |w| \leq d\}$. Let B denote the $D \times 1$ (column) polynomial $B = \sum w \otimes x^w$ and let

$$L = \begin{pmatrix} I & B \\ B^* & I \end{pmatrix}.$$

The proof of the part about compressions to semi-invariant subspaces is a variant of the necessary part of the characterization of NP kernels. (McCullough x2, Agler and McCarthy x2, Jury has most general, sufficiency also add Quiggin.)

3.1 Proof of 3.1

Recall the definition of p in (1). It is easy to see $p_{x_j} \geq 0$ for each j. For instance, if $p_{x_1} < 0$, then consider the tuple $X = (Y, 0, \dots, 0)$ where

$$Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Verify that $tY \in \mathcal{D}c_p$ for all t so that $\mathcal{D}c_p$ is not psuedo-bounded.

Now, let $1 \leq m < d$ denote the smallest integer such that there is a word z with $|z| = m + 1$ and $p_z < 0$. Let $\mathcal{B} = \{w : |w| \leq m\} \cup \{z\}$. Let

$$r = I - \sum_{|w| \in \mathcal{B}} p_w w \tag{2}$$

and let k denote the rational function

$$k(w) = \frac{1}{r(w)} = I + \sum_{n \geq 1} r(w)^n = I + \sum k_v v. \tag{3}$$

Alternatively, for each $v \in \mathcal{B}$ with $|v| > 0$,

$$0 = k_v - \sum \{p_u k_w : uw = v,\ |u| > 0\} \tag{4}$$

so that k_v is defined recursively as

$$k_v = \sum \{p_u k_w : uw = v,\ |w| < |v|\}.$$

Note that (4) implies, as $p_\emptyset = 1$, $p_w \geq 0$ for $|w| \leq m$, and $p_{x_j} > 0$ for each j, $k_v > 0$ for all $|v| \leq m$ and

$$k_z = \sum \{p_u k_w : uw = z,\ 0 < |w| < |v|\} + p_z. \tag{5}$$

We also think of k as a kernel in the NC tuples $\{x_1, \dots, x_g\}$ and $\{y_1, \dots, y_g\}$ as

$$k(y, x) = \sum k_v (y^v)^* x^v,$$

where $x^v = v$ and y^v is the same as x^v but with x_j replaced by y_j; so $y^v = v(y)$.

There are now two cases to consider. In the first case $k_z > 0$ so that the kernel k determines a Hilbert space on finite linear combinations of the words $w \in \mathcal{B}$ by declaring $< v, w >= 0$ if $v \neq w$ and $< v, v >= k_v$ for $v, w \in \mathcal{B}$. There is no need to mod out null vectors because $k_v > 0$ for all $v \in \mathcal{B}$. Define operators S_j on H_k by $S_j x_k v = v$ if $j = k$ and 0 if $j \neq k$ and $S_j \emptyset = 0$, where $\emptyset$ is the empty word and of course $v \in \mathcal{B}$.

Proposition 3.2 *With notations above, $r(S) = I + \sum_{|w| \leq m} p_w w(S)^* w(S) = P_0$, where P_0 is the projection of $\mathcal{M}$ onto the span of $\{\emptyset\}$.*

Given words w and v, say that w *(left) divides* v *with divisor* u if $v = wu$

Proof. Observe that $w(S)^* v = u$ if $v = wu$ and $w(S)^* v = 0$ otherwise. In particular, if $v \neq v'$, then $< w(S)^* v, w(S)^* v' >= 0$. On the other hand, $< w(S)^v, w(S)^* v >$ is either $< u, u >= k_u$ or 0 depending upon whether or not w divides v. Hence, if $v \in \mathcal{B}$, but $v \neq \emptyset$, then

$$< r(S)v, v >= \sum \{p_w k_u : v = wu \ |w| \leq m\}$$

which, by (4) is 0. The result follows. ••

Let $\mathcal{N}$ denote the span of $\{w : w \in \mathcal{B},\ w \neq \emptyset\}$ and observe that $\mathcal{N}$ is semi-invariant for S_j. Let T_j denote the compression of S_j to $\mathcal{N}$. In particular, if x_ℓ divides v, $v = x_\ell w$, but $v \neq x_\ell$, then $T_j v = w$. Otherwise $T_j v = 0$. Consequently,

$$\begin{aligned} < r(T)z, z > &= -\sum \{p_w k_u : z = uw \ u \neq \emptyset\} \\ &= -\sum \{p_w k_u : z = uw\} + p_z \\ &= 0 + p_z < 0 \end{aligned}$$

where (4) was used in the third equality and the assumption $p_z < 0$ in the inequality. Finally, note that, from the definitions, $r(T) = p(T)$ so that $< p(T)z, z >< 0$ and $\mathcal{D}c_p$ is not heavy convex.

We now take up the case $k_z \leq 0$ which is rule out with an argument similar to that above, but contradicting the psuedo-bounded hypothesis rather than the closed with respect to compression to semi-invariant subspace hypothesis.

By (5),

$$0 \leq -(\sum \{p_u k_w : uw = z,\ 0 < |w| < |v|\} + p_z)$$

In particular, the polynomial

$$f(t) = -(\sum \{p_u k_w t^{2|w|} : uw = z,\ 0 < |w| < |v|\} + p_z t^{2|z|})$$

satisfies $f(1) \geq 0$. Given $0 \leq t < 1$, let $\epsilon(t) = |f(t)| + (1-t)$ and define a Hilbert space $H_{k,t}$, much as H_k was defined, by declaring $< v, w >= 0$ if $v, w \in \ mathcalB$ but $v \neq w$, $< v, v >= k_v$ if $|v| \leq m$, and $< z, z >= \epsilon(t)$. As before let $S_j(t)$ denote the backward shift operators on $H_{k,t}$. Observe, for $0 \leq t < 1$ and $|v| \leq m$,

$$< p(tS(t))v, v >= k_v - \sum\{p_w k_u t^{2|w|} : v = wu,\ |w| > 0\}.$$

Thus, as $p_w \geq 0$ and $< p(S)v, v >= 0$ by (3.2), $< p(tS(t))v, v >> 0$ for all $0 \leq t < 1$ and $|v| \leq m$. Further,

$$\begin{aligned} < p(tS(t))z, z > &= \epsilon(t) - \sum\{p_w k_u t^{2|w|} : v = wu,\ m \geq |w| > 0\} - p_z t^{2|z|} \\ &\geq \epsilon(t) - |f(t)| \\ &= 1 - t. \end{aligned}$$

Hence, $< p(tS(t))z, z >> 0$ for $0 \leq t < 1$. It follows that $p(tS(t)) > 0$ for all $0 \leq t < 1$ Write $z = x_\ell y$ and note $< tS_\ell(t)z, z >= t < y, y >= k_y$. Thus, as $< z, z >= \epsilon(t)$ and as t tends to 1, ϵ tends to 0, $tS(t)$ is unbounded as t tends to 1. Thus, $\mathcal{D}c_p$ is not psuedo-bounded. This completes the proof of part (a) of (3.1).

To prove part (b) of (3.1), in view of part (a) and the fact that heavy convex implies closed with respect to compressions to semi-invariant subspaces, without loss of generality, it may be assumed that $p_w \geq 0$ for each $|w| > 0$ and $p_w > 0$ for each $|w| = 1$. Also, as the degree of p is d, there exists a word z such that $|z| = d$ and $p_z > 0$. By relabeling if necessary, we assume that $z = x_1 y$. Let $D = \sum_{n=1}^{d-1} g^n$ and view $\mathbb{C}^D$ as the Hilbert space with orthonormal basis $\{w : w \text{ is a word }, 0 < |w| < d\}$. Given a tuple $X = (X_1, \ldots, X_g)$ define $MNCP_*$ polynomials as follows. Let $B = \sum_{0<|w|<d} w \otimes x^w$, $E = y \otimes x_1^*$ and

$$L = \begin{pmatrix} I & B & E \\ B^* & I & 0 \\ E^* & 0 & I + EE^* \end{pmatrix}.$$

An application of the Cholesky algorithm gives,

Lemma 3.3 *Let $X = (X_1, \ldots, X_g)$ be a given tuple. Then $p(X) \geq 0$ if and only if $L(X) \geq 0$.*

Proof. In the proof we will encounter

$$\begin{aligned} B^*(X)E(X) &= (\sum_{0<|w|<d} w^* \otimes X^{*w})y \otimes X_1^* \\ &= \sum_w w^* y X^{*w} X_1^* \\ &= X^{*y} X_1^* = X^{*z} \end{aligned} \tag{6}$$

where we have used w^*v is 1 if $v = w$ and 0 otherwise.

Since the $(1,1)$ entry of $L(X)$ is I, $L(X)$ is positive semi-definite if and only if the Schur complement of the $(1,1)$ (block matrix) entry of $L(X)$

$$\begin{pmatrix} I - B(X)^*B(X) & -B^*(X)E(X) \\ -E^*(X)B(X) & I \end{pmatrix} = \begin{pmatrix} I - B(X)^*B(X) & -X_z \\ -X_z^* & I \end{pmatrix}$$

is positive semi-definite. Similarly, the above $(2,2)$ block is positive semi-definite if and only if its Schur complement with respect to its $(2,2)$ entry

$$I - B(X)^*B(X) - X^{*z}X^z = p(X)$$

is positive semi-definite. ••

3.2 Result Complex –ORIGINAL VERSION

SCOTT ?? WE WANT L affine linear? OR $L > -I$ or $L > P$ with P a projection ??

If $L \in MNCP_*$ is affine linear and $X = (X_1, \dots, X_g)$ is a tuple of operators, then $L(X) \geq 0$ is called a *Linear Matrix Inequality (LMI)*. Note if $\mathcal{P}_1$ is a collection of self-adjoint affine linear polynomials from $MNCP_*$, then $\mathcal{D}c_{\mathcal{P}_1}$ is an heavy?? convex positivity domain.

Theorem 3.4 *If $\mathcal{D}c_{\mathcal{P}}$ is a bounded heavy?? convex positivity domain, then there exists a collection $\mathcal{P}_1$ of self-adjoint affine linear polynomials from $MNCP_*$ such that $\mathcal{D}c_{\mathcal{P}_1} = \mathcal{D}c_{\mathcal{P}}$; if $p \in \mathcal{P}$ and $\epsilon > 0$, then $p + \epsilon \in \mathcal{C}_{\mathcal{P}_1}$. Moreover, if $L \in \mathcal{P}_1$ and $\epsilon > 0$, then $L + \epsilon \in \mathcal{C}_{\mathcal{P}}$.*

Of course a $\mathcal{P}_1$ may be an infinite collection with no bound on the size of the matrix polynomials involved, even if $\mathcal{P}$ is finite. *A conjecture is that $\mathcal{P}_1$ is finite if $\mathcal{P}$ consists of a finite number of polynomials?*

Looks like the result should imply a result for the hereditary case since affine linear polynomials in NCP_* are automatically hereditary. Also should be a symmetric version. Simply include $\pm(x_j - x_j^*)$ in $\mathcal{P}$? to make all $X \in \mathcal{D}c_{\mathcal{P}}$ self-adjoint tuples of operators etc.

3.3 Proof of Complex LMI

Let $\mathcal{P}_1$ denote the collection of all affine linear (square) matrix-valued $MNCP_*$ self-adjoint polynomials L such that $L(X) \geq 0$ whenever $X \in \mathcal{D}c_{\mathcal{P}}$. The boundedness hypothesis implies that

$$L_j = \begin{pmatrix} C & x_j \\ x_j^* & C \end{pmatrix}$$

is in $\mathcal{P}_1$, since $L_j(X) \geq 0$ if and only if $\|X_j\| \leq C$, it follows that $\mathcal{D}c_{\mathcal{P}_1}$ is a bounded positivity domain. It is also heavy?? convex. ($\mathcal{P}_1 \subset \mathcal{C}_{\mathcal{P}}$, so $\mathcal{D}c_{\mathcal{P}} \subset \mathcal{D}c_{\mathcal{P}_1}$ is usual?? thing.)

Suppose $Y \in \mathcal{D}c_{\mathcal{P}}$. If $L \in \mathcal{P}_1$, then $L(Y) \geq 0$ by the definition of $\mathcal{P}_1$ and so $Y \in \mathcal{D}c_{\mathcal{P}_1}$. Hence $\mathcal{D}c_{\mathcal{P}} \subset \mathcal{D}c_{\mathcal{P}_1}$.

To prove the converse (reverse inclusion), fix a seperable infinite dimensional Hilbert space $\mathcal{H}$ (so $\mathcal{H}$ is isomorphic to ℓ^2) and let $\mathcal{D}c$ denote the tuples $X =$

$(X_1, \ldots, X_g)$ of operators on $\mathcal{H}$ such that $X \in \mathcal{D}c_{\mathcal{P}}$. Let $\mathbf{X} = \oplus_{\mathcal{D}c} X$ acting on $\mathbf{H} = \oplus_{\mathcal{D}c} H$. Of course, if $P \in \mathcal{C}_{\mathcal{P}}$, then $q(\mathbf{X}) \geq 0$. Conversely, if $q(\mathbf{X}) > 0$, then $P \in \mathcal{C}_{\mathcal{P}}$ which we prove in the following paragraph.

Suppose $T = (T_1, \ldots, T_g)$ is a tuple of bounded operators acting on the Hilbert space $\mathcal{T}$ and $T \in \mathcal{D}c_{\mathcal{P}}$ and assume that $P \in M_{l \times l}(NCP_*)$ (so $\ell \times \ell$ matrix valued). Given a non-zero vector $\gamma = \oplus_1^\ell \gamma_j \in \oplus_1^\ell \mathcal{T}$, to see that $< q(T)\gamma, \gamma >$ is positive , let $\mathcal{T}_\gamma$ denote the smallest reducing subspace for $\{T_1, \ldots, T_g, T_1^*, \ldots, T_g^*\}$ containing γ. The subspace $\mathcal{T}_\gamma$ is seperable so there are two cases. First, if $\mathcal{T}_\gamma$ is infinite dimensional, then $\mathcal{T}_\gamma$ is isomorphic to $\mathcal{H}$ and, up to unitary equivalence, we may assume that T_γ, the restriction of T to $\mathcal{T}_\gamma$, is a member of $\mathcal{D}c_{\mathcal{P}}$. It follows that, as $q(\mathbf{X}) > 0$, one has $q(T_\gamma) > 0$. Thus $< q(T)\gamma, \gamma > = < q(T_\gamma)\gamma, \gamma >$ is positive. If $\mathcal{T}_\gamma$ is finite dimensional, then replace $\mathcal{T}_\gamma$ with $\oplus_1^\infty \mathcal{T}_\gamma$ and T_γ with $\oplus_1^\infty T_\gamma$ and argue as above. In any event, we have $q(T) > 0$. By the Positivestellensatz, we conclude $P \in \mathcal{C}_{\mathcal{P}}$.

Let $\beta(\mathbf{X})$ denote the subspace of $\mathcal{B}(\mathbf{H})$,

$$\beta(\mathbf{X}) = \{l(\mathbf{X}) : l \text{ is a scalar affine linear NC polynomial}\}.$$

Given $Y \in \mathcal{D}c_{\mathcal{P}_1}$ acting on the Hilbert space $\mathcal{Y}$, consider the mapping $\psi : \beta(\mathbf{X}) \mapsto \beta(Y)$ defined by

$$\psi(l(\mathbf{X})) = l(Y).$$

To see that this mapping is well defined and completely positive, it is enough to show for any ℓ, if $L \in M_{l \times l}(\beta(\mathbf{X}))$ and if $L(\mathbf{X}) \geq 0$, then $L(Y) \geq 0$. But, if $L(\mathbf{X}) \geq 0$ and $\epsilon > 0$, then by what was proved above $L + \epsilon \in \mathcal{C}_{\mathcal{P}}$ and thus $L + \epsilon \in \mathcal{P}_1$. We conclude that $L(Y) \geq -\epsilon$ and since $\epsilon > 0$ is arbitrary, $L(Y) \geq 0$.

Since $\beta(\mathbf{X})$ is a self-adjoint subspace of $\mathcal{B}(\mathbf{H})$ containing the identity and $\psi(I) = I$ (ψ is unital) and completely positive on $\beta(\mathbf{X})$, by the Arveson Extension Theorem and the Steinsprings Representation Theorem, there exists an auxiliary Hilbert space K, an isometry $V : \mathcal{Y} \mapsto K$, and representation $\pi : \mathcal{B}(\mathbf{H}) \mapsto \mathcal{B}(K)$, such that $\psi(\cdot) = V^* \pi(\cdot) V$. Thus, for affine linear scalar polynomials l, we have $\psi(l(Y)) = V^* \pi(l(\mathbf{X})) V$. In particular, $Y = V^* \pi(\mathbf{X}) V$ for an isometry V. (Note that in the construction $\mathbf{X}$ doesn't depend upon Y, but π does.)

If $q \in \mathcal{P}$ and $q = \{q_{\alpha,\beta}\} \in M_{l \times l}(NCP_*)$, then

$$q(\pi(\mathbf{X})) = \{\pi(q_{\alpha,\beta}(\mathbf{X}))\} = ??1 \otimes ?? \pi(q(\mathbf{X}))$$

SCOTT ?? tensor 1??

and so $q(\pi(\mathbf{X})) \geq 0$. Thus, $\pi(\mathbf{X}) \in \mathcal{D}c_{\mathcal{P}}$. (?? SCOTT ?? bill no compran?? Again this is really a general principle. Domains closed with respect to direct sums, unital representations (including restrict to reducing subspaces) and, if heavy?? convex, then compression to arbitrary subspaces even.) Thus, by the heavy convex hypothesis, $Y \in \mathcal{D}c_{\mathcal{P}}$. Hence $\mathcal{D}c_{\mathcal{P}} = \mathcal{D}c_{\mathcal{P}_1}$.

If $p \in \mathcal{C}_{\mathcal{P}}$ and $\epsilon > 0$, then $p(X)+\epsilon I > 0$ for all $X \in \mathcal{D}c_{\mathcal{P}_1}$ and therefore, $p+\epsilon \in \mathcal{C}_{\mathcal{P}_1}$. The other statement is almost the same.

4 DUMP -Cutting down to finite dimensions - REAL-ALREADY PUT IN Poss3 DOCUM

For heavy convex positivity domains there is a bound on the dimension of Hilbert spaces needed in the Positivestellensatz.

DEFINE IN MAIN DOC $M_{\infty\times\infty}$?? Here it should mean bded ops on Hilby, thats what we need.

Theorem 4.1 *Let $\mathcal{P}$ be a collection of symmetric polynomials from ?? $M_{\infty\times\infty}(\mathcal{N}_*))$ and suppose $\mathcal{D}_\mathcal{P}$ is a bounded heavy convex positivity domain. If q is a symmetric $\ell \times \ell$ matrix-valued $M_{l\times l}(\mathcal{N}_*))$ polynomial of degree d and if $q \notin \mathcal{C}_\mathcal{P}$, then there exists a Hilbert space H of dimension at most $\ell \sum_0^d (2g)^n$, a non-zero vector $\gamma \in H$, and a tuple $X = (X_1, \ldots, X_g)$ of operators on H such that $X \in \mathcal{D}_\mathcal{P}$, but $< q(X)\gamma, \gamma > \leq 0$.*

4.1 Proof of Finite I

From the Positivestellensatz, there exists a tuple $X = (X_1, \ldots, X_g)$ acting on a Hilbert space H and a non-zero vector $\gamma = \oplus\gamma_j \in \oplus_1^\ell H$ such that $< q(X)\gamma, \gamma >$. Here $q \in M_{l\times l}(\mathcal{N}_*)$ and has degree d. Let

$$\mathcal{M} = \operatorname{span}\{w(X)\gamma_j : w \text{ is a word of length at most } d\}.$$

Then $\mathcal{M}$ has dimension at most $\ell \sum_0^d (2g)^n$ and if P is the projection onto $\mathcal{M}$, then $< q(PXP)\gamma, \gamma > = < p(X)\gamma, \gamma > \leq 0$. On the other hand, $PXP \in \mathcal{D}_\mathcal{P}$ by the heavy convex hypothesis.

5 Cutting down to finite dimensions- ORIGINAL

For heavy convex positivity domains there is a bound on the dimension of Hilbert spaces needed in the Positivestellensatz.

Theorem 5.1 *Let $\mathcal{P}$ be a collection of self-adjoint polynomials from ?? $MNCP_*$?? and suppose $\mathcal{D}c_{\mathcal{P}}$ is a bounded heavy convex positivity domain. If q is a self-adjoint $\ell \times \ell$ matrix-valued $MNCP_*$ polynomial of degree d and if $q \notin \mathcal{C}_{\mathcal{P}}$, then there exists a Hilbert space H of dimension at most $\ell \sum_0^d (2g)^n$, a non-zero vector $\gamma \in H$, and a tuple $X = (X_1, \ldots, X_g)$ of operators on H such that $X \in \mathcal{D}c_{\mathcal{P}}$, but $< q(X)\gamma, \gamma > \leq 0$.*

5.1 Proof of Finite I

From the Positivestellensatz, there exists a tuple $X = (X_1, \ldots, X_g)$ acting on a Hilbert space H and a non-zero vector $\gamma = \oplus \gamma_j \in \oplus_1^\ell H$ such that $< q(X)\gamma, \gamma >$. Here $q \in M_{l \times l}(NCP_*)$ and has degree d. Let

$$\mathcal{M} = \text{span}\{w(X)\gamma_j : w \text{ is a word of length at most } d\}.$$

Then $\mathcal{M}$ has dimension at most $\ell \sum_0^d (2g)^n$ and if P is the projection onto $\mathcal{M}$, then $< q(PXP)\gamma, \gamma > = < p(X)\gamma, \gamma > \leq 0$. On the other hand, $PXP \in \mathcal{D}c_{\mathcal{P}}$ by the heavy convex hypothesis.

6 DUMP- convex defs done much better in Poss4.tex

A positivity domain $\mathcal{D}c_{\mathcal{P}}$ is **compression convex** or **heavy convex** if $PXP = (PX_1P, \ldots, PX_gP) \in \mathcal{D}c_{\mathcal{P}}$ whenever $X = (X_1, \ldots, X_g) \in \mathcal{D}c_{\mathcal{P}}$ and P is a projection.

Seemingly closer to the notion of convex is the condition: if X and Y are tuples and $C_1^T C_1 + C_2^T C_2 = I$, then

$$\begin{aligned} C_1^T X C_1 + C_2^T Y C_2 &:= (C_1^T X_1 C_1, \ldots, C_1^T X_g C_1) + (C_2^T Y_1 C_2, \ldots, C_2^T Y_g C_2) \qquad (7) \\ &= C_1^T X_1 C_1 + C_2^T Y_1 C_2, \cdots, C_1^T X_g C_1 + C_2^T Y_g C_2) \end{aligned}$$

is in $\mathcal{D}c_{\mathcal{P}}$.

———

SAME AS ordinary convex, plus closed wrt unitary equiv, compression to reducing spaces.

0 in set not needed.

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Lemma 6.1 *A set $\mathcal{D}c_{\mathcal{P}}$ in $\mathcal{C}_{\mathcal{P}}^{\ell}$ which contains 0 satisfies condition (7) if and only if it satisfies the projected convexity condition.*

Proof Assume that condition (7) is satisfied. Consider a projection P. Set $C_1 = P$ and $C_2 = 1 - P$, and pick X and 0 both tuples in $\mathcal{C}_{\mathcal{P}}^{\ell}$. Condition (7) implies PXP is in $\mathcal{D}c_{\mathcal{P}}$.

Assume projected convexity. Given C_1, C_2 construct an isometry $V : R^{2\ell} \to R^{\ell}$ by

$$V := \begin{pmatrix} C_1^T & C_2^T \end{pmatrix}.$$

Now

$$V(X \oplus Y)V^T = C_1^T X C_1 + C_2^T Y C_2.$$

This gives, since $X, Y \in \mathcal{D}c_{\mathcal{P}}$ implies $X \oplus Y$ is in $\mathcal{D}c_{\mathcal{P}}$, that $C_1^T X C_1 + C_2^T Y C_2$ is in $\mathcal{D}c_{\mathcal{P}}$.

NEED TO SET stuff above carefully -now it is nonsense.?? NEED IN B(HILBY)??

7 DUMP- Old real arveson

————————————————————————-

OLD

SPECULATION:

Call (in sync with SCV) a real β subspace of a complex Hilbert space **totally real** provided that $\beta \cap i\beta$ contains only 0.

SPEC 1 H is real Hilbert space. Suppose ψ is a map on subspace β of $B(H)$ into $B(\tilde{H})$, where $B(H)$ is a totally real subspace of $B(Hc)$, and $B(\tilde{H})$ is a totally real subspace of $B(\tilde{H}c)$. Then extends to a complex completely positive map

$$\Psi : B(H) \to B(\tilde{H})$$

provided ψ is real completely positive.

SPEC 2

A real completely positive map ψ on a totally real subspace β of $B(H)$ into $B(\tilde{H})$ has a representation as below with homomorphism π on real spaces $B(H) \to B(K)$ And $V : \mathcal{Y} \to \mathcal{K}$.

IF the speculation above holds, THEN the proof for complex should go thru for real.