EIGENVALUES OF BROWNIAN MOTIONS ON $\mathrm{GL}(N,\mathbb{C})$

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Abstract. We prove that the empirical law of eigenvalues of Brownian motion on the Lie Group $\mathrm{GL}(N,\mathbb{C})$ converges almost surely to a deterministic probability measure, characterized by a free stochastic differential equation. This fully resolves a conjecture made by Philippe Biane in 1997. Our analysis includes a family $\{B = B_{\varrho,\zeta} : |\zeta| < \varrho\}$ of nondegenerate diffusion processes on $\mathrm{GL}(N,\mathbb{C})$ whose laws are invariant under unitary conjugation, with initial distributions assumed to be uniformly bounded and invertible.

The crux of our analysis is a strong quantitative approximation of Brownian motion $B(t)$ on $\mathrm{GL}(N,\mathbb{C})$ for small t by a single increment $I + W(t)$, where $W = W_{\varrho,\zeta}$ is an elliptic Brownian motion in the Lie algebra $\mathfrak{gl}(N,\mathbb{C}) = \mathbb{M}_N(\mathbb{C})$. Specifically, for any $t \in [0,1]$ and $\delta > 0$,

$$\mathbb{P}\left(\|B(t) - I - W(t)\| \geq \delta\right) \leq (Ct/\delta)^{N^{2/3}}$$

for a constant $C = C_\varrho$. Leveraging independence of multiplicative increments of the Brownian motion then allows us to use powerful (anti-)concentration tools for Gaussian matrices to complete the Hermitization procedure for convergence of eigenvalues.

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1. Introduction

In this paper, we answer the longstanding question of convergence of eigenvalues of Brownian motion on the Lie group $\mathrm{GL}(N, \mathbb{C})$.

Brownian motion and heat kernel analysis on Lie groups have been studied extensively since the 1970s, cf. [66, 57, 55, 34, 5, 1, 4, 82, 47, 83, 49, 35]. One effective construction of Lie *group* Brownian motion is through a stochastic differential equation driven by "flat" Brownian motion on the Lie *algebra*; see (1.3). This provides a direct connection to the classical ensembles of random matrix theory. The unitary group $\mathrm{U}(N)$ has, as Lie algebra, the space of skew-Hermitian matrices in $\mathbb{M}_N(\mathbb{C})$; the "flat" Brownian motion on this space is $\mathbf{i}X^N(t)$ where the Hermitian matrix $X^N(t)$ is a *Gaussian Unitary Ensemble* (scaled by $\sqrt{t}$). Meanwhile, the general linear group $\mathrm{GL}(N, \mathbb{C})$ has as Lie algebra the full space $\mathbb{M}_N(\mathbb{C})$; the "flat" Brownian motion $W^N(t)$ here is, for each t, a matrix of all i.i.d. complex normal random variables, i.e. a *Ginibre ensemble* (scaled by $\sqrt{t}$). These ensembles comprise two of the most well-known settings for large-N limits of eigenvalue distributions: Wigner [88] proved the density of eigenvalues of $X^N(t)$ converges to the semicircle law, while Ginibre [43] proved that $W^N(t)$'s eigenvalues converge to the circular law.

Remark 1.1. *Wigner's semicircle law was generalized beyond the Gaussian setting to reach essentially its optimal form within a decade; cf. [6, 7, 46]. On the other hand, even conjecturing the optimal form of the circular law took 20 years following Ginibre's work, cf. [44], and another 25 years after that for a full resolution, cf. [81, 20]. This is a testament to the vastly greater difficulty working with eigenvalues of non-normal matrices than selfadjoint matrices.*

As curved-space complements to these classical theorems, it is very natural therefore to study large-N limits of eigenvalues of Brownian motions on $\mathrm{U}(N)$ and $\mathrm{GL}(N, \mathbb{C})$. The unitary case was initiated by Philippe Biane in [17, 18], with some related work by Eric Rains around the same time [77]. He showed that, with the correct scaling of the metrics on the groups $\mathrm{U}(N)$, the unitary Brownian motion $U(t) = U^N(t)$ has a large-N limit $(u(t))_{t \geq 0}$ which he called the *free unitary Brownian motion*. The multiplicative increments $U(s)^{-1}U(t)$, $s < t$, are independent for all N; by comparison, the increments $u(s)^{-1}u(t)$ are *freely* independent. The free unitary Brownian motion quickly became a central and useful object in the theory of free entropy, allowing classical regularizing properties of heat flow to be used in difficult entropy and information problems in free probability, cf. [86, 28] and many, many following papers. Its study, and the study of the convergence $U^N(t) \rightharpoonup u(t)$, progressed steadily over the last two decades with stronger forms of convergence of eigenvalues, including concentration at the edge and absence of outliers [26]; strong metric convergence and mesoscopic scale analysis of eigenvalue evolution [65], linear fluctuations [61], applications to Yang–Mills theory [40, 36], connections with non-intersecting Brownian motions [62], etc.

In parallel, Biane introduced the *free multiplicative Brownian motion* $(b(t))_{t\geq 0}$ – another noncommutative stochastic process with freely independent multiplicative increments. In [18], it played a central role in constructing a free version of the Segal–Bargmann–Hall transform: an analytic transform in geometric quantization theory that concretely implements wave-particle duality. The transform intertwines heat kernels (i.e. marginal distributions of Brownian motions) on $\mathrm{U}(N)$ and its complexification $\mathrm{U}(N)_{\mathbb{C}} = \mathrm{GL}(N, \mathbb{C})$. While connected by analogy with Brownian motion $B^N(t)$ on $\mathrm{GL}(N, \mathbb{C})$, Biane wrote [17, p. 19] "It is very likely that the [free multiplicative Brownian motion] process is the limit in distribution... of the Brownian motion with values in $\mathrm{GL}(N, \mathbb{C})$... but we have not proved this." The techniques Biane invented to prove convergence in the unitary setting are unavailable for $\mathrm{GL}(N, \mathbb{C})$, making the problem much harder for the same reason the circular law is much harder than its semicircular cousin: most matrices in $\mathrm{GL}(N, \mathbb{C})$ are non-normal, so the spectral theorem is no use. (Indeed, in [59, Prop. 4.15], it was shown that, with probability 1, $B^N(t)$ is not normal at any time $t > 0$ for $N > 1$.)

The framework to understand the large-N limit of $B^N(t)$ was created independently in [24] and [39]: namely, the Laplacian on $\mathrm{GL}(N, \mathbb{C})$, when applied to trace polynomial functions, has an explicit form $\mathcal{D} + \frac{1}{N^2}\mathcal{L}$ where $\mathcal{D}$ is a first-order pseudo-differential operator and $\mathcal{L}$ is a second-order linear differential operator. Together with concentration bounds, this led to proof of the almost sure convergence of the process $B^N(t)$ to $b(t)$ in terms of $*$-distributions: I.e. convergence of traces of polynomial functions of the process at different times. This was generalized in [59, 60] to a two-parameter family of diffusion processes on $\mathrm{GL}(N, \mathbb{C})$, all Brownian motions with respect to invariant metrics, which includes the unitary Brownian motion as a boundary case. In [25], linear fluctuations around the limit were studied. Meanwhile, the original Segal–Bargmann–Hall transform was extended to allow for complexified time [41], and this gave rise to a third (and final, in a sense explained below) parameter for invariant metrics. Very recently [11] upgraded the convergence $B^N(t) \rightharpoonup b(t)$ to this 3-parameter "elliptic" setting, and moreover upgraded the convergence from traces to norms of poynomial functions (so-called strong convergence).

All that being said, the connection between polynomial functions of matrix entries and eigenvalues of that matrix is very tenuous in the non-normal case, and so Biane's original conjecture about full convergence of the Brownian motion on $\mathrm{GL}(N, \mathbb{C})$ was left open. Until now. Our main Theorem 1.6 proves convergence of the density of eigenvalues of all invariant Brownian motions on $\mathrm{GL}(N, \mathbb{C})$.

To set the stage for this main result, we step back briefly to concretely describe Brownian motions on fairly general Lie groups.

1.1. **Brownian Motion on (Unimodular) Lie Groups.**

Let $\mathfrak{G}$ be a Lie group (of dimension ≥ 1), with Lie algebra $\mathfrak{g} = T_I G$, the tangent space at the identity $I \in \mathfrak{G}$. Given any choice of inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$, there is a unique left-invariant Riemannian metric on $\mathfrak{G}$ which coincides with the inner product at I (defined by dragging vectors at some point $A \in \mathfrak{G}$ back to I by the differential of the left multiplication map $L_{A^{-1}}$); we use the same notation for the metric and the inner product at I. The **Brownian motion** $B = (B(t))_{t\geq 0}$ on the Riemannian manifold $(\mathfrak{G}, \langle\ ,\ \rangle)$ is the Feller–Markov process whose infinitesemial

generator is $\frac{1}{2}\Delta_{(\mathfrak{G},\langle\ ,\ \rangle)}$; i.e. $\frac{1}{2}$ the Laplace–Beltrami operator for the given metric. As usual, this defines the transition semigroup for the process for any given initial distribution on $\mathfrak{G}$.

If $\mathfrak{G}$ is a unimodular group, the operator $\Delta_{(\mathfrak{G},\langle\ ,\ \rangle)}$ can be effectively computed as follows. Fix any orthonormal basis β of $\mathfrak{g}$; then

$$\Delta_{(\mathfrak{G},\langle\ ,\ \rangle)} = \sum_{\xi\in\beta} \partial_\xi^2 \tag{1.1}$$

where, for $f \in C^2(\mathfrak{G})$, $(\partial_\xi f)(A) = \frac{d}{dt} f(A\exp(t\xi))\big|_{t=0}$. (It is a standard calculus exercise that the sum-of-squares operator on the right-hand-side of (1.1) does not depend on which orthonormal basis β is used.) Here, $\exp\colon \mathfrak{g} \to \mathfrak{G}$ is the exponential map which is a local diffeomorphism from a neighborhood of $0 \in \mathfrak{g}$ onto a neighborhood of $I \in \mathfrak{G}$; if $\mathfrak{G}$ is a matrix Lie group, exp is simply the matrix exponential.

The sum-of-squares form for the Laplacian allows for a simple description of the Brownian motion in terms of a stochastic differential equation driven by Brownian motion in the Lie algebra. Specifically: for a given orthonormal basis β of $\mathfrak{g}$, let $\{W_\xi\colon \xi\in\beta\}$ be independent standard Brownian motions on $\mathbb{R}$, and define

$$W(t) = W_{(\mathfrak{g},\langle\ ,\ \rangle)}(t) := \sum_{\xi\in\beta} W_\xi(t)\,\xi \tag{1.2}$$

(again, a standard exercise shows that the law of the process W does not depend on which orthonormal basis is used). Then the Brownian motion satisfies the Stratonovich SDE

$$B(t) = B(0) + \int_0^t B(s)\circ dW(s). \tag{1.3}$$

In the case the $\mathfrak{G}$ is a matrix Lie group, there is a simple conversion between the Stratonovich and Itô integrals:

$$B(t) = B(0) + \int_0^t B(s)\,dW(s) + \frac{1}{2}\int_0^t B(s)\Xi\,ds \tag{1.4}$$

where $\Xi = \sum_{\xi\in\beta}\xi^2$, cf. [64, p. 116]. (The Itô correction term is $\frac{1}{2}$ the quadtratic covariation of B and W.) This sum-of-squares matrix Ξ (which, again, does not depend on the orthonormal basis β) is in the center of $\mathfrak{g}$.

Remark 1.2. *The SDE (1.4) makes sense and holds true for all Lie groups, not just matrix Lie groups, provided one makes sense of Ξ as the Casimir operator in the universal enveloping algebra of $\mathfrak{g}$; such considerations are beyond what is necessary for our present interests.*

Provided $B(0)$ is invertible and independent from W, (1.4) immediately implies that $B_I := B(0)^{-1}B$ satisfies the same SDE with $B_I(0) = I$; hence, Brownian motion with any given initial distribution B_0 can equivalently be constructed as $(B_0 B(t))_{t\ge 0}$ where B is a Brownian motion started at I (i.e. satisfying (1.4) with $B(0) = I$) independent from B_0. Similar considerations show that B has *independent multiplicative increments*: for $0 \le s < t$, $B(s)^{-1}B(t)$ is independent from $\{B(r)\colon r \le s\}$. Moreover, the inverse is itself an Itô process: a stochastic calculus exercise shows that the process $\mathcal{I}(t)$ defined by

$$\mathcal{I}(t) = I - \int_0^t dW(s)\circ\mathcal{I}(s) = I - \int_0^t dW(s)\mathcal{I}(s) + \frac{1}{2}\int_0^t \Xi\mathcal{I}(s)\,ds \tag{1.5}$$

is the inverse of $B(t)$ for each t; i.e. $B(t)\mathcal{B}(t) = I$. That is: since $-W$ has the same law as W, the inverse $B(t)^{-1} = \mathcal{B}(t)$ is a *right*-invariant Brownian motion on $\mathfrak{G}$, driven by W. See [59, Section 4.2] for further discussion.

1.2. **Brownian Motions on** GL($N, \mathbb{C}$)**.**

Now, let us specialize to the case $\mathfrak{G} = \mathrm{GL}(N, \mathbb{C})$, in which case the Lie algebra is $\mathfrak{g} = \mathfrak{gl}(N, \mathbb{C}) = \mathbb{M}_N(\mathbb{C})$, the algebra of all $N \times N$ complex matrices. The above constructions are determined by a choice of inner product on $\mathbb{M}_N(\mathbb{C})$; in other words, (left-invariant Riemannian) Brownian motions on $\mathrm{GL}(N, \mathbb{C})$ are parametrized by the symmetric space of positive definite $N \times N$ matrices. Note that the sum-of-squares matrix Ξ can be computed directly from the given $\mathfrak{gl}(N, \mathbb{C})$ Brownian motion W (1.2) by

$$\mathbb{E}[W(t)^2] = \sum_{\xi_1, \xi_2 \in \beta} \mathbb{E}[W_{\xi_1}(t) W_{\xi_2}(t)] \xi_1 \xi_2 = \sum_{\xi \in \beta} t\xi^2 = t\Xi. \tag{1.6}$$

We fix one reference metric, the (scaled) **Hilbert–Schmidt inner product**

$$\langle A, B \rangle_N := N \operatorname{Re} \operatorname{Tr}(B^* A) \tag{1.7}$$

where $\operatorname{Tr}(A) = \sum_{j=1}^N A_{jj}$ is the trace. (The scaling factor N is the right choice for a large-N bulk limits of eigenvalues, as explained below.) The Lie algebra Brownian motion W corresponding to inner product (1.7) is the **Ginibre Brownian motion**: for each $t > 0$, the matrix $W(t)$ is a Ginibre ensemble whose entries are all independent complex Gaussian random variables each of variance $\frac{t}{N}$. It can be decomposed in the form $W = X + \mathbf{i}Y$ where X, Y are independent Hermitian Brownian motions of size N (cf. Definition 2.10). A calculation shows that $\mathbb{E}[W(t)^2] = 0$ in this case, so $\Xi = 0$. Hence, the associated Brownian motion B on $\mathrm{GL}(N, \mathbb{C})$ satisfies SDE (1.4) with no Itô correction term: $dB(t) = B(t)\, dW(t)$. This is sometimes called the **standard Brownian motion** on $\mathrm{GL}(N, \mathbb{C})$; in the case $N = 1$ it is known as (complex) exponential Brownian motion.

In this paper, we consider the general class of Brownian motions on $\mathrm{GL}(N, \mathbb{C})$ driven by **elliptic Brownian motions** on $\mathbb{M}_N(\mathbb{C})$: i.e.

$$W = e^{\mathbf{i}\theta}(aX + \mathbf{i}bY) \tag{1.8}$$

where X and Y are independent Hermitian Brownian motions (cf. Definition 2.10), and $a, b, \theta \in \mathbb{R}$ with $ab \neq 0$. Equation (1.8) is over-paramatrized: $a \mapsto \pm a$ and $b \mapsto \pm b$ do not change the law of the process; moreover since X, Y are exchangeable, the map $(\theta, a, b) \mapsto (\theta + \frac{\pi}{2}, -b, a)$ also preserves the law of W. We may therefore restrict $b \geq a > 0$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}]$. It is useful to make the following change of variables:

$$\begin{aligned} \varrho &:= \mathbb{E}\operatorname{tr}_N[|W(1)|^2] = a^2 + b^2 \\ \zeta &:= \mathbb{E}\operatorname{tr}_N[W(1)^2] = e^{2\mathbf{i}\theta}(a^2 - b^2). \end{aligned} \tag{1.9}$$

where $\operatorname{tr}_N = \frac{1}{N}\operatorname{Tr}$ is the normalized trace on $\mathbb{M}_N(\mathbb{C})$ and, for any matrix A, $|A|^2 := A^* A$. For the parameter ranges above in the case $a \neq b$, the map $(\theta, a, b) \mapsto (\varrho, \zeta)$ is one-to-one onto the domain $\{(\varrho, \zeta) \colon 0 < |\zeta| < \varrho\}$, with inverse given by

$$\theta = \frac{1}{2} \arg \zeta \qquad a = \sqrt{\frac{1}{2}(\varrho - |\zeta|)} \qquad b = \sqrt{\frac{1}{2}(\varrho + |\zeta|)}.$$

In the case $\zeta = 0$, $W = e^{\mathbf{i}\theta}a(X + \mathbf{i}Y)$ is $e^{\mathbf{i}\theta}a$ times a Ginibre Brownian motion, identically distributed for all θ. Hence all elliptic Brownian motions (1.8) are parametrized (in law) by (ϱ, ζ) with $|\zeta| < \varrho$; we henceforth denote $W = W_{\varrho,\zeta}$ when appropriate. The Ginibre Brownian motion is the special case $W_{1,0}$.

Remark 1.3. *Note that the law of the process $(W(t))_{t\geq 0}$ is the same as the law of the process $(W(rt)/\sqrt{r})_{t\geq 0}$ for any $r > 0$. In particular, the collection of random matrix laws $W_{\varrho,\zeta}(t)$ is over-parametrized, since*

$$W_{\varrho,\zeta}(t) \stackrel{\mathrm{d}}{=} W_{t\varrho,t\zeta}(1) \stackrel{\mathrm{d}}{=} W_{1,\zeta/\varrho}(\varrho t). \tag{1.10}$$

In some places (notably Sections 1.6 and 5) the reduced parametrization $W(t, \zeta) = W_{t,\zeta}(1) \stackrel{\mathrm{d}}{=} W_{1,\zeta/t}(t)$ will be used.

Note that all the elliptic Brownian motions (1.8) are invariant under unitary conjugation: if $U \in \mathrm{U}(N)$ is a deterministic unitary matrix then $UW_{\varrho,\zeta}U^*$ is equal in distribution to $W_{\varrho,\zeta}$. Unitary conjugation invariance is a historically useful property for connecting random matrices to large-N limits with free probability, cf. Proposition 2.7. In [41, Theorem 3.3, Proposition 3.7], the authors characterized all inner products, and hence all Brownian motion processes W (1.2) on $\mathfrak{gl}(N, \mathbb{C})$, that are $\mathrm{U}(N)$-conjugation invariant; they are all of the form $W_{\varrho,\zeta} + \frac{1}{\sqrt{N}}W'I$ where W' is a standard complex Brownian motion (i.e. 2-dimensional for all N). This decaying diagonal rank-2 perturbation is negligible in the large-N limit; hence, the elliptic Brownian motions $\{W_{\varrho,\zeta} \colon \varrho > 0, |\zeta| < \varrho\}$ asymptotically span all unitarily invariant Brownian motions on $\mathbb{M}_N(\mathbb{C})$.

Since Ξ is in the center of $\mathfrak{gl}(N, \mathbb{C}) = \mathbb{M}_N(\mathbb{C})$, it follows from (1.6) and (1.9) that $\Xi = \mathbb{E}[W(1)^2] = \zeta I$. Hence, the main objects of study in the present work are Brownian motions $B_0B_{\varrho,\zeta}$ where

$$B_{\varrho,\zeta}(t) = I + \int_0^t B_{\varrho,\zeta}(s)\, dW_{\varrho,\zeta}(s) + \frac{\zeta}{2}\int_0^t B_{\varrho,\zeta}(s)\, ds. \tag{1.11}$$

The standard Brownian motion on $\mathrm{GL}(N, \mathbb{C})$ is the specal case $B_{1,0}$, driven by the Ginibre Brownian motion $W_{1,0}$ on $\mathfrak{gl}(N, \mathbb{C})$.

Remark 1.4. *It is necessary for $|\zeta| < \varrho$ in order for the diffusion $W_{\varrho,\zeta}$ to be nondegenerate; but from the formulas, we could also consider a case where $|\zeta| = \varrho$, like $(\varrho, \zeta) = (1, 1)$. This corresponds in (1.9) to $(a, b, \theta) = (0, 1, 0)$, i.e. $W_{1,1} = \mathbf{i}Y$. This Brownian motion is confined to the space of skew-Hermitian matrices, the Lie algebra of $\mathrm{U}(N)$. Now, let $U := B_{1,1}$ be the corresponding multiplicative process (1.11):*

$$U(t) = I + \mathbf{i}\int_0^t U(s)\, dY(s) - \frac{1}{2}\int_0^t U(s)\, ds.$$

Note that $U(t)^*$ then satisfies (1.5), showing that $U(t)^* = U(t)^{-1}$, i.e. U takes values in $\mathrm{U}(N)$. Indeed, this is the unitary Brownian motion whose large-N limit was studied by Biane [17].

As we proceed to consider putative large-N limits, it is useful to disambiguate the dimension of the Brownian motion processes.

Notation 1.5. *The processes above will be indexed by N as needed. Moreover, the parameters (ϱ, ζ) will remain fixed throughout most of this paper. As such,*

for readability, we often suppress the explicit dependence of the notation for the Brownian motion processes on the parameters, and define

$$W_{\varrho,\zeta}(t) = W^N_{\varrho,\zeta}(t) = W^N(t) \in \mathbb{M}_N(\mathbb{C})$$
$$B_{\varrho,\zeta}(t) = B^N_{\varrho,\zeta}(t) = B^N(t) \in \mathrm{GL}(N, \mathbb{C}).$$

1.3. **Main Results.**

Given a matrix $A \in \mathbb{M}_N(\mathbb{C})$, denote its N complex eigenvalues (counted with multiplicity) by $\{\lambda_j(A)\}_{j=1}^N$. The singular values are the square roots of the eigenvalues of A^*A, or equivalently the eigenvalues of $\sqrt{A^*A}$; they are denoted

$$\sigma_{\max}(A) = \sigma_1(A) \geq \cdots \geq \sigma_N(A) = \sigma_{\min}(A).$$

Note that $\sigma_{\max}(A) = \|A\|$ and $\sigma_{\min}(A) = \|A^{-1}\|^{-1}$ (otherwise $\sigma_{\min}(A) = 0$).

The **empirical law of eigenvalues** of A, denoted μ_A, is the random probability distribution with equal masses at all eigenvalues of A:

$$\mu_A = \frac{1}{N}\sum_{j=1}^{N} \delta_{\lambda_j(A)}. \tag{1.12}$$

Our main theorem asserts that almost surely, the empirical law of eigenvalues $\mu_{B^N(t)}$ converges weakly to a deterministic measure. In fact, we prove this more generally for the Brownian motion $B_0^N B^N(t)$ for any uniformly bounded and invertible initial distribution B_0^N that has a large-N limit of its own.

Theorem 1.6. *Let $\varrho > 0$, $\zeta \in \mathbb{C}$ with $|\zeta| < \varrho$, and let $t \geq 0$. Let $B^N(t) = B^N_{\varrho,\zeta}(t)$ denote a Brownian motion on* $\mathrm{GL}(N, \mathbb{C})$*, cf.* (1.11)*.*

For each $N \in \mathbb{N}$, let B_0^N be a random matrix in $\mathrm{GL}(N, \mathbb{C})$ *independent from the process B^N, and suppose that B_0^N converges a.s. in $*$-distribution (cf. Definition 2.4) to an operator b_0. Assume further that for some positive constant $K < \infty$,*

$$K^{-1} \leq \sigma_{\min}(B_0^N) \leq \sigma_{\max}(B_0^N) \leq K \tag{1.13}$$

a.s. for all N sufficiently large. Then a.s. the empirical law $\mu_{B_0^N B^N(t)}$ of eigenvalues of the Brownian motion $B_0^N B^N(t)$ converges weakly to a deterministic measure $\mu_{b_0,t}$ (described in Sections 1.5-1.6 below).

Our proof uses Girko's Hermitization framework (cf. Section 1.5), proving sufficient control on small-to-medium scale singular values of $B_0^N B^N(t) - zI$ for a.e. $z \in \mathbb{C}$ to guarantee convergence of the log potential. They key tool we develop to accomplish these goals is a small-time affine approximation of the process $B^N(t)$. As shown in [16], Brownian motion B (1.4) on a matrix Lie group is well-approximated by a Donsker-like random walk approximation:

$$B^N(t) \approx B^N_n(t) := \prod_{i=1}^{\lfloor t2^n \rfloor} \left(I + W(\tfrac{i}{2^n}) - W(\tfrac{i-1}{2^n}) + \tfrac{1}{2^{n+1}}\Xi\right).$$

In [38] (where the matrix random walk $B^N_n(t)$ was considered but scaled with general k terms in the product not only $k = 2^n$) the authors provided locally time-uniform pathwise L^p bounds for this approximation as $n \to \infty$, but those bounds grow with N. In the present work, we show (in the case of interest, $B = B_{\varrho,\zeta}$ where $\Xi = \zeta I$) that, for a fixed $t \geq 0$, $\|B^N(t) - B^N_n(t)\|_{L^2} \leq C_{n,t} 2^{-n/2}$ with a constant $C_{t,n}$ that is independent of N; moreover, the free multiplicative Brownian motion satisfies the

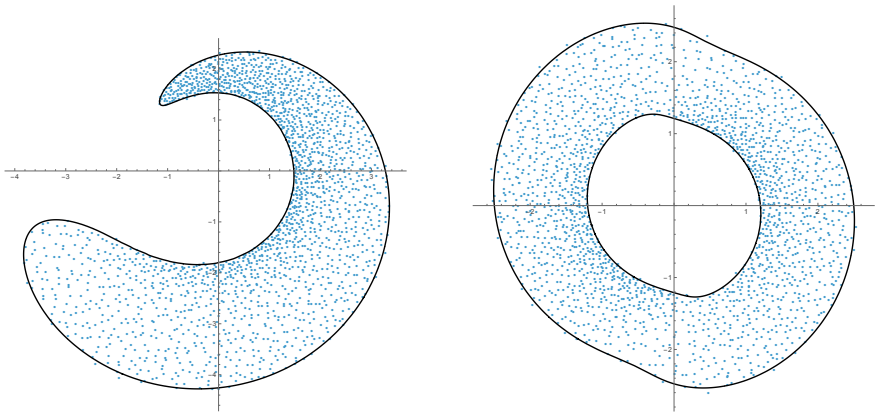

FIGURE 1. Eigenvalues of $B_0^N B^N(1)$ with $N = 2000$ and $(\varrho, \zeta) = (2, 1.7 + 0.6\mathbf{i})$. On the left B_0^N is normal with eigenvalues $1 - \mathbf{i}, 2 + \mathbf{i}, 3 - 3\mathbf{i}$; on the right B_0^N is non-normal

$$B_0^N \cong b_0 = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}.$$

same estimate cf. Proposition 3.3. The increments $(1 + \frac{\zeta}{2k})I + W(\frac{i}{k}) - W(\frac{i-1}{k})$ are independent Gaussian matrices. By careful repeated use of Gaussian integration by parts (aka the Schwinger–Dyson equations, Section 2.3), we compare the product $B_n^N(t)$ to a *single* increment. The main technical result, which is of independent interest, is as follows.

Theorem 1.7. *For any finite T there exists $C = C_{\varrho,T} < \infty$ so that, for all $t \in [0, T]$ and $\delta > 0$,*

$$\mathbb{P}\left(\left\|B^N(t) - I - W^N(t)\right\| \geq \delta\right) \leq \left(\frac{Ct}{\delta}\right)^{N^{2/3}}.$$

Theorem 1.7 is proved in Section 3. The $N^{2/3}$ arises from a moment estimate (Theorem 3.5) which is the culmination of iterated integration by parts and Gronwall's lemma calculation, bounding the expected trace moments

$$\mathbb{E}\left[\mathrm{tr}_N\left(|B^N(t) - I - W^N(t)|^{2k}\right)\right] \leq (tC_{\varrho,T})^{2k} \exp\left\{C_{\varrho,T}\frac{k^4}{N^2}\right\}. \tag{1.14}$$

Taking the $2k$th root of the left-hand-side and sending $k \to \infty$ gives the operator norm; taking the $2k$th root on the right yields t times an exponential of order k^3/N^2. Taking $k \sim N^{2/3}$ and applying Markov's inequality yields Theorem 1.7.

Note that Equation (1.14) is the result of several years of development of a new strategy consisting in interpolating random matrices with their free limit. This strategy was first introduced in [27] before being strongly refined in [74]. Those papers study the case of polynomials in independent GUE random matrices. In particular, in the second paper it was proved that if P_N is a polynomial in GUE random matrices, then its kth moment $\mathbb{E}[\mathrm{tr}_N(P_N^k)]$ for $k \geq 0$ is given by a Taylor series in $\frac{1}{N}$, whose coefficients are built from the powers of an operator L. In light of the approximation $B^N(t) \approx B_n^N(t)$ of $B^N(t)$ by polynomials in GUE random

matrices, the question boils down to continuity estimates for powers of L. We prove the approximation in L^2 sense, and it turns out L is L^2-continuous in the appropriate sense for the main result of [74] to show that the moments of $B_n^N(t)$ converge as $n \to \infty$ to yield estimates on the moments of $B^N(t)$.

Finally, note that the interpolating strategy has been used to study a large array of models. In [72, 73], the author investigated the unitary case, in [76] the case of random matrix models with a potential, in [75] this strategy was even used to study tensors of random matrices which provided a solution, albeit not the first, to the Peterson–Thom conjecture thanks to the work of Ben Hayes in [52]. The case of (ϱ, ζ) Brownian motions was even investigated in [11] although with a different interpolation which does not yield a sufficiently sharp bound for our needs and also does not allow to consider $B^N(t)$ and $W^N(t)$ simultaneously.

To control the small singular values of scalar shifts $B^N(t) - zI$ we use Theorem 1.7 in conjunction with the following observation. If B^N and C^N are independent (ϱ, ζ) Brownian motions then, for any $t > 0$ and any $\varepsilon \in (0, t)$, $B^N(t-\varepsilon)C^N(\varepsilon)$ has the same law as $B^N(t)$. From Theorem 1.7 we have

$$C^N(\varepsilon) \approx I + W^N(\varepsilon) \tag{1.15}$$

for small ε, and hence $B^N(t) \approx B^N(t-\varepsilon)(I + W^N(\varepsilon))$. The robust behavior of singular values of the elliptic Ginibre matrix $W^N(\varepsilon)$ can thence be parlayed into good behavior of singular values for $B^N(t) - zI$. We note that the basic strategy of exploiting more tractable randomness in the Lie algebra was brilliantly used in [78] for the setting of the orthogonal and unitary groups. In that argument the analogue of the step (1.15) is elementary, whereas here we rely on Theorem 1.7, the key technical innovation of this work.

We use this framework (fleshed out in Section 3) to prove Proposition 4.1 below, which gives quantitative lower bounds on the singular values of arbitrary shifts of $B_0^N B^N(t)$, holding with high probability. We also provide a second proof of anti-concentration of mesoscopic scale singular values in Section 5, using the PDE methods pioneered in [37, 50] (for the case of initial condition $B_0^N = I$, but yielding finer regularity). Finally, Section 6 then shows how those estimates are used to prove the requisite convergence in Theorem 1.6; see also Section 1.5 for a bigger picture discussion of the Hermitization procedure we generally follow here.

1.4. **Free Multiplicative Brownian Motions.**

To fully describe the deterministic limit density of eigenvalues in Theorem 1.6, we need some constructs and tools from free probability theory; see Section 2.1 for a brief primer and references.

The Brownian motion $B^N = B^N_{\varrho,\zeta}$ (1.11) on GL($N, \mathbb{C}$) is driven by the elliptic Brownian motion $W^N = W^N_{\varrho,\zeta}$ (1.8), which is constructed from a pair X^N, Y^N of independent Hermitian Brownian motions (Definition 2.10). These processes have a large-N limit. Let $(\mathscr{A}, \tau)$ be a W^*-probability space that supports two freely independent free semicircular Brownian motions x, y (Definition 2.5). Voiculescu showed [85] that $\left(X^N, Y^N\right)$ converges a.s. in finite dimensional $*$-distributions to (x, y) as $N \to \infty$. In particular,

$$w_{\varrho,\zeta} = e^{\mathbf{i}\theta}(ax + \mathbf{i}by) \tag{1.16}$$

is a free stochastic process in $\mathscr{A}$ that is the a.s. large-N limit in finite dimensional $*$-distributions of $W^N_{\varrho,\zeta} = e^{\mathbf{i}\theta}(aX^N + \mathbf{i}bY^N)$; here $(\varrho, \zeta) \leftrightsquigarrow (\theta, a, b)$ as in (1.9). The

process $w_{\varrho,\zeta}$ is a noncommutative martingale [56], so stochastic integrals and quadratic covariation may be treated analogously to the classical setting. In particular, the free Itô equation

$$b_{\varrho,\zeta}(t) = 1 + \int_0^t b_{\varrho,\zeta}(s)\, dw_{\varrho,\zeta}(s) + \frac{\zeta}{2}\int_0^t b_{\varrho,\zeta}(s)\, ds \tag{1.17}$$

has a unique solution in $\mathscr{A}$ for all $t \geq 0$, cf. [19, 58]. The original free multiplicative Brownian motion introduced by Biane in [17, 18] is the special case $(\varrho, \zeta) = (1, 0)$, while the free unitary Brownian motion is the edge case $(\varrho, \zeta) = (1, -1)$. The two-parameter family introduced in [59] corresponds to $b_{\varrho,\zeta}$ with $\zeta \in (-\varrho, \varrho)$ (i.e. real ζ). The fully complex parameter ζ with $|\zeta| < \varrho$ was later introduced in [41] in the context of heat kernel analysis on compact type Lie groups (notably the question of when the heat kernel can be complexified in both space *and time*), yielding the complete family of metrics defining the Brownian motions $B_{\varrho,\zeta}$ on $\mathrm{GL}(N, \mathbb{C})$ that are $\mathrm{U}(N)$-conjugation invariant. The three-parameter families of free stochastic processes $w_{\varrho,\zeta}$ and $b_{\varrho,\zeta}$ were studied extensively in [50, 51], as strong limits in [11], and most recently in terms of general additive perturbations of $w_{\varrho,\zeta}$ in [89].

Notation 1.8. *As with the matrix-valued processes $W^N = W^N_{\varrho,\zeta}$ and $B^N = B^N_{\varrho,\zeta}$, we generally suppress the explicit dependence of the notation for the Brownian motion processes on the parameters (ϱ, ζ), and define*

$$w_{\varrho,\zeta}(t) =: w(t) \qquad b_{\varrho,\zeta}(t) =: b(t).$$

The following lemma guides our aim to describe the limit distribution $\mu_{b_0,t}$ of eigenvalues of $B^N(t)$ in Theorem 1.6.

Lemma 1.9. *Let $\varrho > 0$ and $\zeta \in \mathbb{C}$ with $|\zeta| < \varrho$, and let B^N be a (ϱ, ζ) Brownian motion on $\mathrm{GL}(N, \mathbb{C})$. Let $(\mathscr{A}, \tau)$ be a W^*-probability space that contains a free multiplicative (ϱ, ζ) Brownian motion b, and let b_0 be an operator in $\mathscr{A}$ that is freely independent from b.*

Let B_0^N be a sequence of random matrices in $\mathbb{M}_N(\mathbb{C})$ independent from the process B^N such that B_0^N converges a.s. in $*$-distribution to b_0 as $N \to \infty$.

Then for any $t \geq 0$, $B_0 B^N(t)$ converges a.s. in $*$-distribution to $b_0 b(t)$.

Under the much stronger assumption that B_0^N converges *strongly* to b_0 (see Definition 2.4), the recent paper [11] proved Lemma 1.9 with the stronger conclusion of strong convergence. Lemma 1.9 is proved in Section 2.2.

1.5. **Singular Values, Eigenvalues, and Hermitization.**

Lemma 1.9 suggests that the large-N limit $\mu_{b_0,t}$ of the empirical law of eigenvalues of $B_0^N B^N(t)$ is connected to the free multiplicative Brownian motion $b_0 b(t)$. To understand exactly how, this section is devoted to explaining the connection between shifted singular values and eigenvalues, known as the *Hermitization procedure* introduced in [44], and independently discovered in the operator algebras context in [23]. The connection for each finite N is not robust under large-N limits, as we explain; hence the difficulty of the problem that our main theorem overcomes.

For notational ease, we work with a general (random) matrix ensemble $A^N \in \mathbb{M}_N(\mathbb{C})$. Suppose that $A^N \rightharpoonup a$ converges in $*$-distribution to some operator a in a W^*-probability space $(\mathscr{A}, \tau)$. Moreover, suppose that $\limsup_{N\to\infty} \|A^N\| < \infty$. (If A^N is random, we may interpret there conditions to hold almost surely, or in probability, depending what mode of convergence the conclusion demands.)

As we are interested in eigenvalues of A^N, the matrix $A_z^N := A^N - zI$ ($z \in \mathbb{C}$) will play a role in the following computations. Note that $A_z^N = A^N - zI \rightharpoonup a - z =: a_z$ in $*$-distribution, and therefore

$$\lim_{N\to\infty} \operatorname{tr}_N \left[(A_z^{N\,*} A_z^N)^k \right] = \tau \left[(a_z^* a_z)^k \right] \quad \text{for all } k \in \mathbb{N}. \tag{1.18}$$

Let $\sigma_j(A_z^N)$, $1 \le j \le N$ denote the singular values of A_z^N; that is, $\sigma_j(A_z^N)$ are the eigenvalues of $\sqrt{A_z^{N\,*} A_z^N} =: |A_z^N|$. Let ν_z^N denote the empirical law of singular values of A_z^N, and let ν_z denote the spectral measure of $|a_z|$:

$$\nu_z^N = \frac{1}{N} \sum_{j=1}^N \delta_{\sigma_j(A_z^N)} = \mu_{|A^N - zI|} \qquad \& \qquad \nu_z = \mu_{|a-z|}. \tag{1.19}$$

Then (1.18) says that the even moments of ν_z^N converge a.s. to those of ν_z. By the uniform boundedness assumption on A^N, and the boundedness of $a \in \mathscr{A}$, there is a constant $M = M(z) < \infty$ so that $\{\nu_z^N, \nu_z\}_{N\in\mathbb{N}}$ are all supported in the fixed compact interval $[0, M]$. It therefore follows that $\nu_z^N \rightharpoonup \nu_z$ converges weakly; i.e.

$$\int_0^\infty f \, d\nu_z^N \to \int_0^\infty f \, d\nu_z \quad \forall\, f \in C_b(\mathbb{R}_+). \tag{1.20}$$

Now, the function $f(x) = \log x$ is neither continuous nor bounded on $\mathbb{R}_+$, but let us suspend disbelief and consider using it as a test function anyway.

On the left-hand-side of (1.20), note that

$$\begin{aligned} \int_0^\infty \log \, d\nu_z^N = \frac{1}{N} \sum_{j=1}^N \log \sigma_j(A_z^N) = \frac{1}{N} \log \left[\prod_{j=1}^N \sigma_j(A_z^N) \right] \\ = \frac{1}{2N} \log \det[A_z^{N\,*} A_z^N] \end{aligned} \tag{1.21}$$

where the first equality is the definition of ν_z^N and the last equality is the realization of the determinant as the product of the eigenvalues. Noting that the determinant is multiplicative and also that $\det A^* = \overline{\det A}$, it follows that this equals

$$\begin{aligned} \frac{1}{N} \log \left| \det[A_z^N] \right| &= \frac{1}{N} \log \left| \det[A^N - zI] \right| \\ &= \frac{1}{N} \sum_{j=1}^N \log |z - \lambda_j(A^N)| \end{aligned} \tag{1.22}$$

where $\lambda_j(A)$ are the eigenvalues of the matrix A, and the last equality is the realization/definition of the eigenvalues as the roots of the characteristic polynomial.

Combining (1.21) and (1.22) with (1.12) defining the empirical law of eigenvalues μ_{A^N} of A^N shows that

$$\int_0^\infty \log \, d\nu_z^N = \int_{\mathbb{C}} \log |z - \lambda| \, \mu_{A^N}(d\lambda) =: 2\pi L_{A^N}(z). \tag{1.23}$$

That is: up to a factor of 2π, the log integral of ν_z^N is the **log potential** L_{A^N} of the empirical measure μ_{A^N}. This is significant because the log potential of any probability measure on $\mathbb{C}$ is a subharmonic function whose Laplacian is the measure itself: i.e. $\Delta_z L_\mu(z) = \mu$ in distribution sense. (This is easy to see specifically for μ_{A^N} directly from (1.22). The function $z \mapsto \log|\lambda - z|$ is harmonic on $\mathbb{C} \setminus \{\lambda\}$, and is the Coulomb potential in the plane, so its Laplacian is a point mass at λ.)

Turning to the right-hand-side of (1.20), we analogously define

$$L_a(z) := \frac{1}{2\pi}\int_0^\infty \log\, d\nu_z = \frac{1}{2\pi}\int_0^\infty \log\, d\mu_{|a-z|}. \tag{1.24}$$

In [23], Brown showed that L_a is a subharmonic function on $\mathbb{C}$ (taking values in $[-\infty, \infty)$) whose space Laplacian ΔL_a is a probability measure on $\mathbb{C}$, which we denote μ_a, the **Brown measure** of a. To be precise: the Brown measure μ_a is the unique probability measure on $\mathbb{C}$ satisfying

$$\int_{\mathbb{C}} \psi(z)\, \mu_a(dz) = \int_{\mathbb{C}} \Delta\psi(z)\, L_a(z)\, dz \qquad \forall\, \psi \in C_c^\infty(\mathbb{C}). \tag{1.25}$$

It is a substitute for the empirical law of eigenvalues for more general operators in W^*-probability spaces.

Remark 1.10. (1.23) *and* (1.24) *are fully consistent: treating a matrix A^N as a noncommutative random variable in $(\mathbb{M}_N(\mathbb{C}), \mathrm{tr}_N)$, its log potential as in* (1.24) *is precisely given by* (1.23)*, as the above calculations show. In other words, for a matrix $A^N \in \mathbb{M}_N(\mathbb{C})$, the Brown measure μ_{A^N} is exactly its empirical law of eigenvalues as in* (1.12)*. Brown [23] also showed that, for normal a, the Brown measure μ_a coincides with the spectral measure μ_a.*

The above procedure is known as **Hermitization**. In summary:

(1) Find an operator a in a W^*-probability space so that $A^N \rightharpoonup a$ in $*$-distribution.
(2) Prove that the weak convergence (1.20) extends to allow the test function $f = \log$; i.e. prove that the log potentials $L_{A^N}(z)$ converge pointwise to $L_a(z)$, for (Lebesgue) almost every $z \in \mathbb{C}$.
(3) Use local integrability properties of $(z, \lambda) \mapsto \log|z-\lambda|$ to prove that $L_{A^N} \to L_a$ uniformly on compact subsets of $\mathbb{C}$.
(4) Therefore conclude using (1.25) that $\mu_{A^N} = \Delta L_{A^N} \rightharpoonup \Delta L_a = \mu_a$, i.e. the empirical law of eigenvalues of A^N converges weakly to the Brown measure of a.

The key steps (1) and (2) must be verified (usually with great difficulty) for each particular ensemble A^N under consideration; the remaining steps are then routine.

The Hermitization procedure was introduced by Girko [44] as a method to prove the Circular Law. Ginibre [43] showed that (what we therefore now call) the Ginibre Ensemble (i.e. the $\mathfrak{gl}(N, \mathbb{C})$ Brownian motion $W_{1,0}$ at time 1) has eigenvalues whose empirical measure converges to the uniform density on the unit disk in $\mathbb{C}$. Girko conjectured that this holds for *any* ensemble of random matrices A^N with all independent centered L^2 entries of variance $\frac{1}{N}$, and offered the Hermitization procedure as a roadmap to prove it (without verifying (2)). It took more than 25 years for his program to reach full fruition, with the first major breakthrough by Bai [8] requiring the entries of A^N to have more than 6 finite moments, then numerous improvements culminating with Tao and Vu's [81].

The same Hermitization roadmap has been used for various generalizations of the Circular Law (see the excellent survey [20] for an extensive summary of work up to ∼2011), the much larger class of "Single Ring" ensembles (that are unitarily bi-invariant) [48, 14] and generalizations, e.g. [15, 12, 54], the elliptic law [68], inhomogeneous circular laws [30, 31, 2, 3], local circular laws [21], polynomials of independent matrices [71, 70, 45, 32], sparse directed graphs [29, 13, 63, 79]

and most recently (related to the present work) very general matrix random walks with unitarily bi-invariant increments [38]. The tools needed to prove step (2), i.e. uniform integrability of log against the empirical law ν_z^N of singular values of $A^N - zI$, vary significantly by the particular model in question, and can be quite challenging. Judging from the computations relating ν_z^N with μ_{A^N}, one should expect to require additional fine control on the decay rate of small singular values of $A^N - zI$. It is important to note that this log integrability, and convergence of the empirical law of eigenvalues, *can* fail even for some relatively simple matrix ensembles, cf. [20, Example 1.2].

We conclude this section by restating Theorem 1.6 with the precise identification of the limit measure. Given that Lemma 1.9 verifies step (1), the limit empirical measure of eigenvalues $\mu_{b_0,t}$ should be the Brown measure of the limit $b_0 b(t)$. This is indeed the case.

Theorem (Full statement of Theorem 1.6)**.** *Let $\varrho > 0$, $\zeta \in \mathbb{C}$ with $|\zeta| < \varrho$, and let $t \geq 0$. Let $B^N(t) = B^N_{\varrho,\zeta}(t)$ denote a Brownian motion on $\mathrm{GL}(N, \mathbb{C})$, cf. (1.11), and let $b(t) = b_{\varrho,\zeta}(t)$ be its $*$-distribution limit: a free multiplicative Brownian motion (1.17) in a W^*-probability space $(\mathscr{A}, \tau)$.*

For each $N \in \mathbb{N}$, let B_0^N be a random matrix in $\mathrm{GL}(N, \mathbb{C})$ independent from the process B^N, and suppose that B_0^N converges a.s. in $*$-distribution (cf. Definition 2.4) to an operator $b_0 \in \mathscr{A}$. Assume that (1.13) holds; i.e. for some positive constant $K < \infty$

$$K^{-1} \leq \sigma_{\min}(B_0^N) \leq \sigma_{\max}(B_0^N) \leq K$$

a.s. for all N sufficiently large. Then a.s. the empirical law $\mu_{B_0^N B^N(t)}$ of eigenvalues of the Brownian motion $B_0^N B^N(t)$ converges weakly to the Brown measure $\mu_{b_0,t} := \mu_{b_0 b(t)}$ of the free multiplicative Brownian motion $b_0 b(t)$.

1.6. **The Brown(ian) Measure.**

Theorem 1.6 proves that the empirical law of eigenvalues of Brownian motion $B^N_{\varrho,\zeta}(t)$ on $\mathrm{GL}(N, \mathbb{C})$ with a very general invertible initial state B_0^N converges to an identifiable deterministic measure: the Brown measure of the operator $b_0 b_{\varrho,\zeta}(t)$. These Brown measures (at least for some initial conditions) have been well-studied in the past five years, cf. [37, 53, 50, 33, 42, 51]. These papers succeeded in computing the Brown measure precisely for general $|\zeta| < \varrho$ and any *unitary* initial condition, using stochastic, complex analytic, and PDE methods.

In this section, we briefly summarize known results about these Brown measures (all from 2022 or later), with some pictures of their support sets with simulations of eigenvalues of the associated Brownian motions for fairly large N. Given the now known (thanks to Theorem 1.6) robust connection to the eigenvalues, the simulations are suggestive of the more general picture of the support of the Brown measure for non-unitary initial conditions.

We begin with the case of the "standard" free multiplicative Brownian motion $b_{1,0}$, with unitary initial condition $b_0 = u_0$. The computation of the Brown measure with $b_0 = 1$ was initially accomplished by Driver–Hall–Kemp in [37], and then extended to include unitary initial conditions by Ho–Zhong in [53].

Theorem 1.11 ([37],[53])**.** *Let $b_{0,1}(t)$ be a free multiplicative Brownian motion, and let u_0 be a unitary operator freely independent from $b_{0,1}$. Denote by μ_{u_0} the spectral measure of u_0, a probability measure on the unit circle $\mathbb{T}$.*

Define $T(u_0, \cdot)\colon \mathbb{C} \to \mathbb{R}_+$ *by*

$$T(u_0, z) = \frac{\log|z|^2}{|z|^2 - 1} \left(\int_{\mathbb{T}} \frac{\mu_{u_0}(d\xi)}{|\xi - z|^2} \right)^{-1}. \tag{1.26}$$

Also, for each $t > 0$, *define the open precompact set* $\Sigma(u_0, t)$ *by*

$$\Sigma(u_0, t) = \{z \in \mathbb{C}\colon T(u_0, z) < t\}. \tag{1.27}$$

The Brown measure $\mu_{u_0 b_{1,0}(t)}$ *has support equal to* $\overline{\Sigma(u_0, t)}$. *The Brown measure is absolutely continuous with respect to Lebesgue measure, and has density on* $\overline{\Sigma(u_0, t)}$ *that is real analytic in* $\Sigma(u_0, t)$. *The density has the form* $z = re^{i\theta} \mapsto w_t(\theta)/r^2$, *with the radial function* w_t *bounded by* $w_t(\theta) < \frac{1}{\pi t}$.

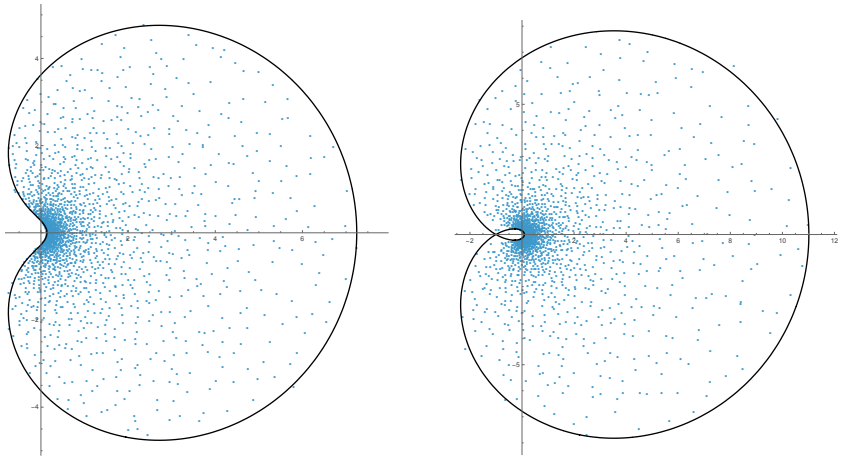

FIGURE 2. Eigenvalues of $B^N_{1,0}(t)$ with $N = 2000$ and $t = 3$ (left) and $t = 4$ (right). Also plotted is the boundary $\partial\Sigma(1, t)$, cf. (1.27).

Following this, in [50] Hall–Ho refined the PDE analysis from the above-mentioned papers to allow the Brownian motions $b_{\varrho,\zeta}(t)$ from the full parameter range $|\zeta| < \varrho$.

Notation 1.12. *The overparametrization of laws of* $W_{\varrho,\zeta}(t)$ *highlighted in* (1.10) *applies equally well to the Brownian motions* $B_{\varrho,\zeta}(t)$. *I.e.*

$$B_{\varrho,\zeta}(t) \overset{\mathrm{d}}{=} B_{t\varrho,t\zeta}(1) \overset{\mathrm{d}}{=} B_{1,\zeta/\varrho}(\varrho t).$$

The same therefore applies to the large-N limit $b_{\varrho,\zeta}(t) \overset{\mathrm{d}}{=} b_{t\varrho,t\zeta}(1) \overset{\mathrm{d}}{=} b_{1,\zeta/\varrho}(\varrho t)$.

[50, 51] elected to simplify parameters and only work with $b_{\varrho,\zeta}(1)$; *we do the same for the remainder of this section. We also follow [50]'s convention and, in this parameter reduction, let* t *play the role of* ϱ. *Hence we treat the operators*

$$B^N(t, \zeta) := B^N_{t,\zeta}(1)$$
$$b(t, \zeta) := b_{t,\zeta}(1).$$

Then the Brownian motion $u_0 b_{1,0}(t)$ *in Theorem 1.11 has the same law as*

$$u_0 b_{1,0}(t) \overset{\mathrm{d}}{=} u_0 b(t, 0).$$

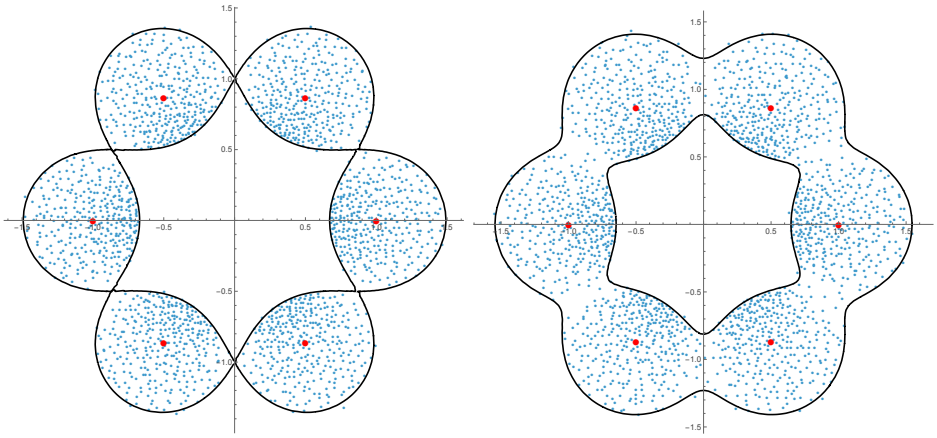

FIGURE 3. Eigenvalues of $B_0^N B_{1,0}^N(t)$ with $N = 2000$ and $t = \frac{2}{3}$ (left) and $t = 0.7$ (right), with initial condition B_0^N a unitary u_6 with equal mass eigenvalues at the 6th roots of unity (highlighted in the figures). Also plotted is the boundary $\partial\Sigma(u_6, t)$, cf. (1.27).

For each $t > 0$ and any $\zeta \in \mathbb{C}$ with $|\zeta| < t$, the Brown measure $\mu_{u_0 b(t,\zeta)}$ is a push-forward of the Brown measure $\mu_{u_0 b(t,0)}$ from Theorem 1.11 under a continuous map that is a diffeomorphism in the interior $\Sigma(u_0, t)$ and a homeomorphism on the closure. To properly define this map, we introduce the following analytic transform: for any operator a in a W^*-probability space, define

$$J_a(z) = \frac{1}{2} \int_{\mathbb{C}} \frac{\xi + z}{\xi - z} \, \mu_a(\xi) \tag{1.28}$$

where μ_a is the Brown measure of a. This integral defines a conformal map on the complement of $\operatorname{Supp} \mu_a$; on this region, it can be written in the form $J_a(z) = \frac{1}{2}\tau[(a + z)(a - z)^{-1}]$. The integral (1.28) may be well-defined for (some) z in the support of μ_a, as will be the case relevant to the following theorem.

Theorem 1.13 ([50, 51])**.** *Let $t > 0$ and $\zeta \in \mathbb{C}$ with $|\zeta| < t$. Let $b(t, \zeta)$ be a free multiplicative Brownian motion cf. Notation 1.12, and let u_0 be a unitary operator freely independent from $b(t, \zeta)$. Define the map $\Psi_{t,\zeta}^{u_0}$ by*

$$\Psi_{t,\zeta}^{u_0}(z) = z \exp\{\zeta J_{u_0 b(t,0)}(z)\}. \tag{1.29}$$

This map is continuous on $\mathbb{C}$. Restricted to $\Sigma(u_0, t)$, it is a diffeomorphism onto its image, an open set we denote $\Sigma(u_0, t, \zeta)$. The map $\Psi_{t,\zeta}^{u_0}$ is a homeomorphism from the closure of $\Sigma(u_0, t)$ onto the closure of $\Sigma(u_0, t, \zeta)$, and is a conformal map between the complements of these closures.

The Brown measure of $u_0 b(t, \zeta)$ is the push-forward under $\Psi_{t,\zeta}^{u_0}$ of the Brown measure of $u_0 b(t, 0)$ (from Theorem 1.11). Its support is $\overline{\Sigma(u_0, t, \zeta)}$, and it has a smooth density on its support.

One key feature of Theorem 1.13 which explains the parameter choices is that, for fixed $t > 0$, the support regions $\{\Sigma(u_0, t, \zeta)\}_{|\zeta|<t}$ are all diffeomorphic to the region $\Sigma(u_0, t)$ from the standard Brownian motion $u_0 b(t, 0)$. I.e. the topology of

the domain where the eigenvalues cluster can change with t, while ζ is deformation parameter that does not change the topology of the support set.

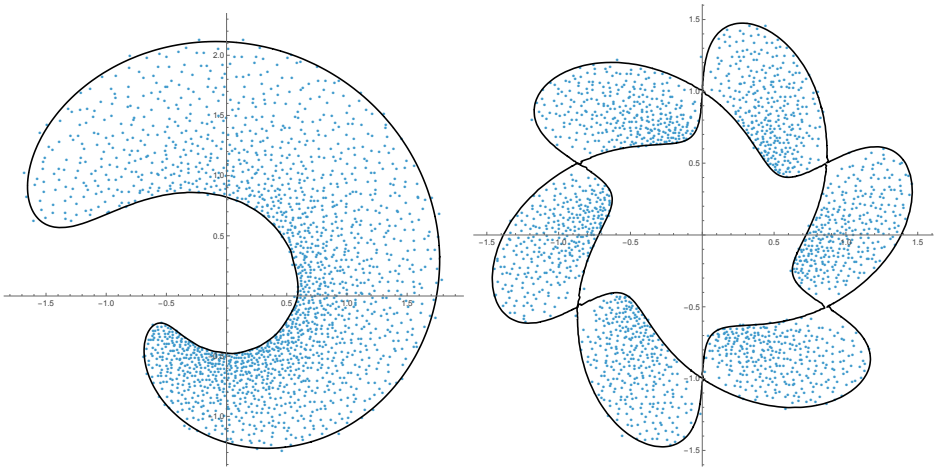

FIGURE 4. Eigenvalues of $B_0^N B^N(t,\zeta)$ with $N = 2000$. On the left, $B_0^N = I$ and $(t,\zeta) = (3, 2-\mathbf{i})$; on the right, B_0^N is unitary with 6th roots of unity as eigenvalues, and $(t,\zeta) = (2/3, -\mathbf{i}/3)$. Also plotted are the boundaries $\partial\Sigma(b_0,t,\zeta)$, cf. Theorem 1.13.

Finally, moving beyond the case of unitary initial conditions, the papers [33, 42] use a related but qualitatively different PDE approach to analyze the support of the Brown measure of $b_0 b(t,\zeta)$ in the case that b_0 is a positive definite operator. In [33], Demni–Hamdi considered the case $\zeta = 0$ and restricted b_0 to be a projection; refining those ideas, Eaknipitsari–Hall [42] considered general $b(t,\zeta)$ and general positive definite b_0. The formulas and techniques in the latter paper are not particularly specific to the positive case, and so we state the results as a special case of a conjecture.

Conjecture 1.14 ([42])**.** *Let b_0 be an invertible operator in a W^*-probability space $(\mathscr{A},\tau)$. Define*

$$\tilde{p}_0(z) = \tau[|b_0 - z|^{-2}] \qquad \& \qquad \tilde{p}_2(z) = \tau[b_0^* b_0 |b_0 - z|^{-2}]$$

where, as usual, for any operator $a \in \mathscr{A}$, $|a|^2 := a^* a$. Define $T(b_0,\cdot)\colon \mathbb{C} \to \mathbb{R}_+$ by

$$T(b_0,z) = \frac{\log(\tilde{p}_2(z)) - \log(|z|^2 \tilde{p}_0(z))}{\tilde{p}_2(z) - |z|^2 \tilde{p}_0(z)}. \tag{1.30}$$

For each $t > 0$, define the open precompact set $\Sigma(b_0,t)$ by

$$\Sigma(b_0,t) = \{z \in \mathbb{C}\colon T(b_0,z) < t\}. \tag{1.31}$$

Let $b(t,\zeta)$ be a free multiplicative Brownian motion (cf. Notation 1.12) freely independent from b_0. Then the support of the Brown measure of $b_0 b(t,\zeta)$ is equal to the image $\overline{\Sigma(b_0,t,\zeta)}$ of $\overline{\Sigma(b_0,t)}$ under the map $\Psi_{t,\zeta}^{b_0}$ of (1.29).

The expression 1.31 is defined by limits if any of the terms diverge. The quantities $\tilde{p}_0(z)$ and $\tilde{p}_2(z)$ are positive but possibly infinite; if one is infinite, a calculus exercise shows that $T(b_0,z) = 0$, meaning that $z \in \Sigma(b_0,t)$ for all $t > 0$.

If b_0 is unitary then $\tilde{p}_2(z) = \tilde{p}_0(z)$, and (1.30) reduces (1.26); so Conjecture 1.14 follows from Theorem 1.13 in the case of a unitary initial condition. In [42], the authors considered the case that $b_0 > 0$ (i.e. positive definite), and the conjecture was half proved in that case: $\operatorname{Supp} \mu_{b_0 b(\varrho,\zeta)}$ was shown to be *contained* in $\overline{\Sigma(b_0, \varrho, \zeta)}$ (with an exception at 0). The obstructions to extending this containment to more general initial conditions were technical, and it is natural to expect it to hold true in full generality. The following simulations of the region $\Sigma(b_0, \varrho, \zeta)$ with eigenvalues of $B^N_{\varrho,\zeta}(1)$ for fairly large N are convincing evidence for the full conjecture.

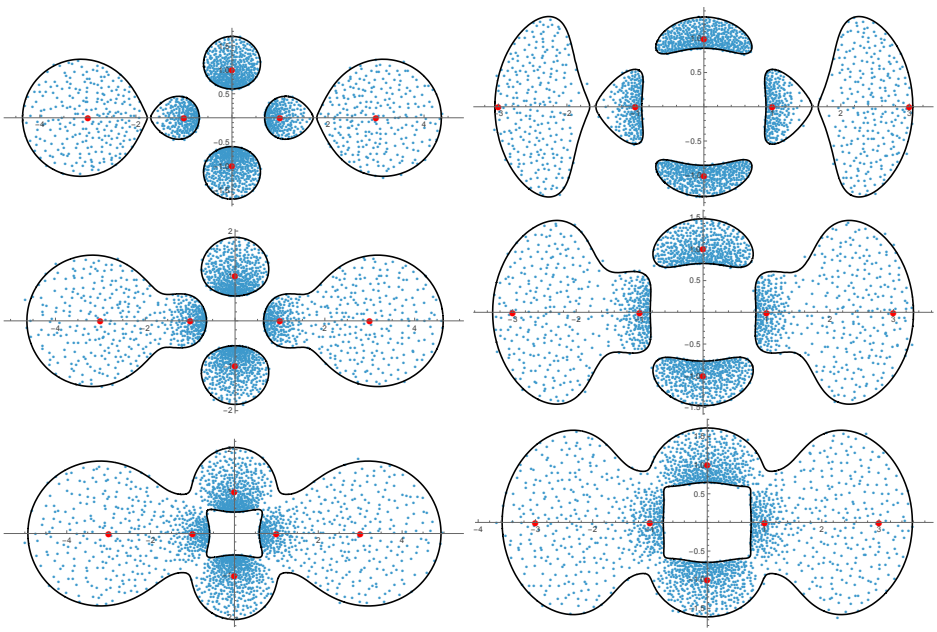

FIGURE 5. Eigenvalues of $B_0^N B^N(t, \zeta)$ with $N = 2000$. From top to bottom, t ranges through $0.8, 1, 1.2$; $\zeta = 0$ on the left and $\zeta = 0.5$ on the right. In all cases, B_0^N is a normal matrix with empirical eigenvalue distribution $\frac{1}{8}(\delta_1 + \delta_{-1} + \delta_3 + \delta_{-3}) + \frac{1}{4}(\delta_{\mathbf{i}} + \delta_{-\mathbf{i}})$. Also plotted are the boundaries $\partial\Sigma(b_0, t, \zeta)$ from (1.31).

2. Preliminaries and Background

2.1. Basic Constructions from Free Probability.

In order to be self-contained, we begin by reminding the following definitions and basic results from noncommutative (and especially free) probability. To begin, we set some basic notation for matrix computations.

Notation 2.1. *The following notation will be used throughout this paper.*

- $\mathbb{M}_N(\mathbb{C})$ *is the set of matrices of size* N.
- Tr_N *is the trace on* $\mathbb{M}_N(\mathbb{C})$*:* $\operatorname{Tr}_N[A] = \sum_{i=1}^N A_{i,i}$*, whereas* $\operatorname{tr}_N = \frac{1}{N}\operatorname{Tr}_N$ *is the normalized trace.*
- $E_{i,j}$ *is the matrix whose only non-zero coefficient is a* 1 *in the* (i, j)*-entry. The size of the matrix* $E_{i,j}$ *will depend on the context.*

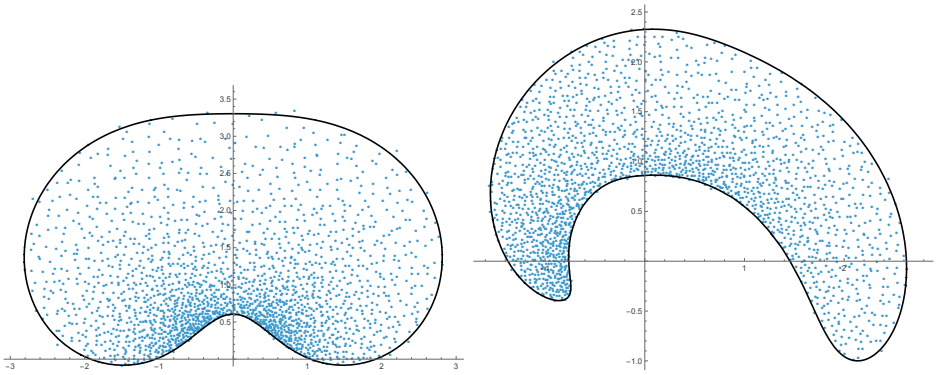

FIGURE 6. Eigenvalues of $B_0^N B^N(1,\zeta)$ with $N = 2000$. The initial condition is equivalent to the non-normal 2×2 matrix

$$B_0^N \cong b_0 = \begin{bmatrix} 1+\mathbf{i} & 1 \\ 0 & -1+\mathbf{i} \end{bmatrix}.$$

On the left, $\zeta = 0$; on the right, $\zeta = 0.5 + 0.5\mathbf{i}$. Also plotted are the boundaries $\partial\Sigma(b_0, 1, \zeta)$ from (1.31).

The algebra $\mathbb{M}_N(\mathbb{C})$ of matrices can be viewed as a noncommutative analog of the algebra of bounded random variables on a probability space, with the normalized trace tr_N playing the role of the expectation. In fact, both L^∞ and $\mathbb{M}_N(\mathbb{C})$ are special cases of a larger framework.

Definition 2.2. *A W^*-probability space $(\mathscr{A}, \tau)$ is a von Neumann algebra (a weakly closed unital algebra of bounded operators on a Hilbert space), equipped with a linear functional $\tau\colon \mathscr{A} \to \mathbb{C}$ that is*

- *a state: $\tau(1_{\mathscr{A}}) = 1$,*
- *tracial: $\tau(ab) = \tau(ba)$ for all $a, b \in \mathscr{A}$,*
- *faithful: $\tau(a^*a) > 0$ for $a \neq 0$, and*
- *normal: τ is continuous in the σ-weak operator topology.*

Some prominent examples of W^*-probability spaces relevant to this paper are

- $(L^\infty(\Omega, \mathscr{F}, \mathbb{P}), \mathbb{E})$: the algebra of L^∞ random variables on a probability space, equipped with the expectation state,
- $(\mathbb{M}_N(\mathbb{C}), \mathrm{tr}_N)$: the algebra of complex $N \times N$ matrices equipped with the normalized trace tr_N.
- The "product" of the last two examples: the algebra $(L^\infty(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{M}_N(\mathbb{C}))$ of matrix-valued bounded (entrywise) random variables, equipped with expectation state $\mathbb{E} \circ \mathrm{tr}_N$. (It will often be useful to instead consider random variables valued in the space $(\mathbb{M}_N(\mathbb{C}), \mathrm{tr}_N)$, so that questions of trace/norm convergence are considered almost surely or in probability.)

Based on the first example, we call the elements of $\mathscr{A}$ (noncommutative) random variables. Note that, as $\mathscr{A}$ is a von Neumann algebra, if $x = x^* \in \mathscr{A}$ is selfadjoint, the spectral theorem yields a unique element $f(x) \in \mathscr{A}$ for any Borel measurable function $f\colon \mathbb{R} \to \mathbb{C}$. In particular, for any $a \in \mathscr{A}$, the operator a^*a is positive and so $|a|^p := (a^*a)^{p/2}$ is a positive operator in $\mathscr{A}$.

Associated to a W^*-probability space $(\mathscr{A}, \tau)$ are (noncommutative) L^p-norms:

$$\|a\|_p := (\tau[|a|^p])^{1/p}, \qquad 1 \leq p < \infty.$$

Note, by the trace property, it follows that $\tau[(a^*a)^{p/2}] = \tau[(aa^*)^{p/2}]$, so that $\|a\|_p = \|a^*\|_p$.

The closure of $\mathscr{A}$ in $\|\cdot\|_p$ is the noncommutative L^p-space $L^p(\mathscr{A}, \tau)$. The state τ extends to a linear functional on $L^1(\mathscr{A}, \tau)$, which we also denote τ. It satisfies the triangle inequality $|\tau[a]| \leq \tau[|a|]$ and the Dominated Convergence Theorem (this is precisely what normal continuity for τ means). The L^p-norms satisfy Hölder's inequality: if $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, and if $a \in L^p(\mathscr{A}, \tau)$ and $b \in L^{p'}(\mathscr{A}, \tau)$, then $ab \in L^1(\mathscr{A}, \tau)$, and

$$|\tau(ab)| \leq \|ab\|_1 \leq \|a\|_p \|b\|_{p'}.$$

In particular, since $\|1_{\mathscr{A}}\|_p = 1$ for all p, it follows that $\|a\|_p^p = \tau[|a|^p \cdot 1_{\mathscr{A}}] \leq \||a|^p\|_r \|1_{\mathscr{A}}\|_{r'} = \|a\|_{pr}^p$ for $r \geq 1$; i.e. $p \mapsto \|a\|_p$ is increasing, so that $L^p(\mathscr{A}, \tau) \subseteq L^q(\mathscr{A}, \tau)$ for $q \leq p$.

Because τ is faithful, it follows that the operator norm on $\mathscr{A}$ satisfies $\|a\| = \lim_{p\to\infty} \|a\|_p$; so we may set $\|a\|_\infty := \|a\|$ and extend Hölder's inequality fully to $1 \leq p \leq \infty$, identifying $L^\infty(\mathscr{A}, \tau) = \mathscr{A}$.

Remark 2.3. *It is worth noting that the limit may be restricted to even integers, so that*

$$\|a\| = \lim_{k\to\infty} \|a\|_{2k} = \lim_{k\to\infty} \left(\tau[(a^*a)^{2k}]\right)^{1/2k}.$$

Computing the norm as a limit of roots of moments will be useful in several cases throughout this paper.

Definition 2.4. *Let $\mathbf{a} = (a_1, \ldots, a_k)$ be a k-tuple of noncommutative random variables in some W^*-probability space $(\mathscr{A}, \tau)$. The* $*$**-distribution** *of $\mathbf{a}$ is the linear functional $\mu_{\mathbf{a}} : P \mapsto \tau[P(\mathbf{a}, \mathbf{a}^*)]$ on the set of polynomials in $2k$ noncommuting indeterminates.*

If $(\mathscr{A}_N, \tau_N)$ is a sequence of W^*-probability spaces for $N \geq 1$, and $\mathbf{a}_N \in \mathscr{A}_N$ are k-tuples, we say $\mathbf{a}_N$ **converges in** $*$**-distribution to** $\mathbf{a}$ *if the $*$-distributions $\mu_{\mathbf{a}_N}$ converges pointwise to $\mu_{\mathbf{a}}$; i.e. if, for all polynomials P in $2k$ noncommuting indeterminates,*

$$\lim_{N\to\infty} \tau_N[P(\mathbf{a}_N, \mathbf{a}_N^*)] = \tau[P(\mathbf{a}, \mathbf{a}^*)].$$

We say that $\mathbf{a}_N$ **converges strongly to** $\mathbf{a}$ *if, in addition to converging in $*$-distribution, also*

$$\lim_{N\to\infty} \|P(\mathbf{a}_N, \mathbf{a}_N^*)\| = \|P(\mathbf{a}, \mathbf{a}^*)\|.$$

If the elements in $\mathscr{A}_N$ are themselves classically random (i.e. a_j^N is actually an $\mathscr{A}_N$-valued random variable), it will be convenient to assume all a_j^N are sampled from the same probability space, in which case we may talk about almost sure *convergence in $*$-distribution or strong convergence. Typically we speak of convergence in probability of the $*$-distribution. This will be most relevant for us with $(\mathscr{A}_N, \tau_N) = (\mathbb{M}_N(\mathbb{C}), \mathrm{tr}_N)$ — i.e. convergence of matrix $*$-distributions or strong convergence in probability or a.s.*

One important special case is for normal operators in $\mathscr{A}$. If $a \in \mathscr{A}$ is normal, i.e. $aa^* = a^*a$, then the Spectral Theorem asserts that there is a projection-valued

measure E^a (taking values in the projections in $\mathscr{A}$) such that $a = \int_{\mathrm{spec}(a)} \lambda \, E^a(d\lambda)$. In the case of a matrix, this measure is finitely-supported on the eigenvalues of a, and $E^a(\{\lambda\})$ is the projection onto the eigenspace of $\lambda \in \mathrm{spec}(a)$. In general, this yields functional calculus: if $f\colon \mathrm{spec}(a) \to \mathbb{C}$ is a measurable function, we may define $f(a) := \int_{\mathrm{spec}(a)} f(\lambda) \, E^a(d\lambda)$.

In this case of a normal operator, the $*$-distribution is fully captured by the **spectral measure** $\mu_a = \tau \circ E^a$. Indeed, the Spectral Theorem yields

$$\int f \, d\mu_a = \int f(\lambda) \tau \circ E^a(d\lambda) = \tau \left[\int f(\lambda) \, E^a(d\lambda) \right] = \tau[f(a)]$$

and taking $f(\lambda) = \lambda^n \bar{\lambda}^m$ we have

$$\tau[a^n a^{*m}] = \int \lambda^n \bar{\lambda}^m \, \mu_a(d\lambda).$$

(When a is normal, all polynomials in a and a^* reduce to linear combination of the terms $a^n a^{*m}$.)

For a selfadjoint bounded operator x, the spectral measure μ_x is supported in the compact set $\mathrm{spec}(x) \subset \mathbb{R}$; hence, μ_x is determined by its moments. In other words: the spectral measure μ_x is the unique compactly-supported measure on $\mathbb{R}$ whose moments are

$$\int_{\mathbb{R}} t^n \, \mu_x(dt) = \tau[x^n], \qquad n \in \mathbb{N}. \tag{2.1}$$

If A^N is a normal matrix in $(\mathbb{M}_N(\mathbb{C}), \mathrm{tr}_N)$, a standard exercise shows that its spectral measure is precisely its empirical law of eigenvalues:

$$\mu_{A^N} = \sum_{j=1}^{N} \delta_{\lambda_j(A^N)}. \tag{2.2}$$

We now come to the framework that gives free probability its name.

Definition 2.5. *Let $(\mathscr{A}, \tau)$ be a W^*-probability space, and let $\mathscr{A}_1, \ldots, \mathscr{A}_n$ be $*$-subalgebras of $\mathscr{A}$, having the same unit as $\mathscr{A}$. They are said to be* **freely independent** *or just* **free** *if, for all k, for all $a_i \in \mathscr{A}_{j_i}$ such that $j_1 \neq j_2$, $j_2 \neq j_3$, $\ldots$, $j_{k-1} \neq j_k$:*

$$\tau\Big((a_1 - \tau(a_1))(a_2 - \tau(a_2)) \ldots (a_k - \tau(a_k))\Big) = 0. \tag{2.3}$$

I.e. products of centered elements from adjacent-alternating algebras have trace 0. Families of noncommutative random variables are said to be free if the $*$-subalgebras they generate are free.

Freeness does not occur in finite-dimensional matrix algebras, or classical L^∞ probability settings, except in fairly degenerate settings (e.g. constants are always freely independent from all other random variables). However, a plethora of examples from random matrix theory exhibit *asymptotic* freeness, i.e. as dimensions of the matrices tend to ∞, they become free. We state a classical example of this phenomenon below.

Definition 2.6. *A* **Haar Unitary Ensemble** *U^N is a random $N \times N$ matrix whose distribution is the Haar measure on $\mathrm{U}(N)$. In a W^*-probability space $(\mathscr{A}, \tau)$, a* **Haar unitary operator** *$u \in \mathscr{A}$ is a unitary operator whose $*$-istribution is determined by $\tau[u^n] = \tau[u^{*\,n}] = \delta_{n0}$ for all $n \in \mathbb{N}$.*

Proposition 2.7 ([84]; [80]). *Let A_0^N, A_1^N be random matrix ensembles, i.e. random elements in the W^*-probability space $(\mathbb{M}_N(\mathbb{C}), \mathrm{tr}_N)$. Let $(\mathscr{A}, \tau)$ be a W^*-probability space, with elements $a_0, a_1 \in \mathscr{A}$ such that $A_0^N \rightharpoonup a_0$ a.s. in $*$-distribution and $A_1^N \rightharpoonup a_1$ a.s. in $*$-distribution. For each $N \in \mathbb{N}$, let U^N be a Haar Unitary Ensemble independent from $\{A_0^N, A_1^N\}$, and let u be a Haar unitary operator in $\mathscr{A}$ freely independent from $\{a_0, a_1\}$. Then $(A_0^N, U^N A_1^N U^{N*})$ converges a.s. in joint $*$-distribution to (a_0, ua_1u^*); ua_1u^* has the same distribution as a_1; and a_0, ua_1u^* are freeely independent. I.e. A_0^N and $U^N A_1^N U^{N*}$ are a.s. asymptotically free.*

Remark 2.8. *Proposition 2.7 is a vital insight connecting random matrix theory sand free probability; it has been a standard way to "generate" asymptotic freeness for three decades. It was originally conceived by Voiculescu in [84], where he proved it in the case that A_0^N, A_1^N are deterministic, and the $*$-distributions of the conjugated matrices are measured in $(\mathbb{M}_N(\mathbb{C}), \mathbb{E}\,\mathrm{tr}_N)$, i.e.* averaged *asymptotic freeness. In [80], Speicher upgraded the theorem to a.s. convergence of the random $*$-distributions, still for deterministic A_0^N, A_1^N. From there, it is a straightforward Tonnelli argument to prove the full statement above; see [67, Theorem 4.9] and the following discussion.*

Some important recent work [9, 22] has expanded the horizons of asymptotic freeness to show it can be achieved in many situations without conjugation by some kind of random rotation/permutation.

Now we introduce important families of free random variables that are large-N limits of Gaussian random matrix ensembles and stochastic processes.

Definition 2.9. *A family of noncommutative random variables $x = (x_1, \ldots, x_n)$ in a W^*-probability space $(\mathscr{A}, \tau)$ is called a* **free semicircular system** *if $x_1, \ldots, x_m$ are freely independent, selfadjoint ($x_i = x_i^*$, $i = 1 \ldots m$), and for all $k \in \mathbb{N}$ and $i = 1, \ldots, m$ one has*

$$\tau(x_i^k) = \int t^k d\sigma(t), \tag{2.4}$$

with $d\sigma(t) = \frac{1}{2\pi}\sqrt{4 - t^2}\ \mathbf{1}_{|t| \le 2}\, dt$ the semicircle distribution. For an explicit construction of such operators, we refer to [69, Lecture 7].

A **free semicircular Brownian motion** *$(x(t))_{t \ge 0}$ is a family of semicircular variables such that $x(t)/\sqrt{t}$ has the semicircle distribution for each $t > 0$, and the additive increments of x are stationary and freely independent: for $0 \le t_1 < t_2 < \infty$, $x(t_2) - x(t_1)$ has the same law as $x(t_2 - t_1)$, and $x(t_2) - x(t_1)$ is freely independent from $(x(t))_{0 \le t \le t_1}$. See [19] for a detailed construction and properties.*

We then define classical random matrix models.

Definition 2.10. *A* **Hermitian Brownian motion** *$(X^N(t))_{t \ge 0}$ of size N is the Brownian motion on the vector space of Hermitian matrices in $\mathbb{M}_N(\mathbb{C})$, with respect to the scaled Hilbert–Schmidt inner product (1.7). Entry-wise, it can be described as follows:*

- *For $1 \le i \le N$, the random variables $\{\sqrt{N}[X^N(t)]_{i,i}\}_{t \ge 0}$ are independent standard Brownian motions on $\mathbb{R}$.*
- *For $1 \le i < j \le N$, the random variables $\{\sqrt{2N}\,\mathrm{Re}[X^N(t)]_{i,j}\}_{t \ge 0}$ and $\{\sqrt{2N}\,\mathrm{Im}[X_t^N]_{i,j}\}_{t \ge 0}$ are independent standard Brownian motions on $\mathbb{R}$, independent from $\{[X^N(t)]_{i,i}]\}_{t \ge 0}$.*

A random matrix Y^N whose law is equal to that of $X^N(1)$ is called a **Gaussian Unitary Ensemble**, *or* GUE *for short; its law can be described by its density with respect to Lebesgue measure on Hermitian matrices, which is proportional to $X \mapsto \exp(-N\mathrm{Tr}_N[X^2])$. This density is invariant under unitary conjugation, hence the name. The process $(X^N(t))_{t\geq 0}$ is sometimes called a* GUE *Brownian motion.*

Definition 2.11. *A* **Ginibre Brownian motion** *of size N is a $\mathbb{M}_N(\mathbb{C})$-valued diffusion process Z^N of the form*

$$Z^N \overset{\mathrm{d}}{=} \frac{1}{\sqrt{2}}(X^N + \mathbf{i}Y^N)$$

where X^N, Y^N are independent Hermitian Brownian motions of size N. Entrywise, this can be described simply: for $1 \leq i, j \leq N$, $\{\sqrt{2N}\,\mathrm{Re}Z^N_{i,j}, \sqrt{2N}\,\mathrm{Im}Z^N_{i,j}\}_{t\geq 0}$ are all independent standard Brownian motions on $\mathbb{R}$.

The random matrix $Z^N(1)$, whose entries are all i.i.d. complex normal random variables, is called a **Ginibre Ensemble** *first introduced in [43]. Equivalently, the law of Z has a density with respect to Lebesgue measure on $\mathbb{M}_N(\mathbb{C})$ which is proportional to $Z \mapsto \exp(-N\mathrm{Tr}_N[Z^*Z])$. Note that this density is invariant under $Z \mapsto UZV^*$ for any deterministic unitary matrices $U, V \in \mathrm{U}(N)$; thus, the law of the Ginibre ensemble is unitarily bi-invariant.*

Hermitian Brownian motions have large-N limits as stochastic processes, in the following strong sense.

Proposition 2.12 ([85])**.** *Let $\{\mathbf{X}^N(t) = (X_1^N(t), \ldots, X_m^N(t))\}_{t\geq 0}$ be independent Hermitian Brownian motions of size N. Let $\{\mathbf{x}(t) = (x_1(t), \ldots, x_m(t))_{t\geq 0}$ be freely independent free semicircular Brownian motions. Then for any $t_1, \ldots, t_k \geq 0$, the km-tuple $(\mathbf{X}^N(t_1), \ldots, \mathbf{X}^N(t_k))$ converges a.s. to the km-tuple $(\mathbf{x}(t_1), \ldots, \mathbf{x}(t_k))$ in $*$-distribution as $N \to \infty$. I.e. $\mathbf{X}^N$ converges in finite-dimensional $*$-distributions to $\mathbf{x}$.*

We will often need to consider free semicircular variables and matrices in the same space. To do so, let $(x_{a,b})_{1\leq a\leq b\leq N}$ and $(y_{a,b})_{1\leq a<b\leq N}$ be free semicircular variables in a W^*-probability space $(\mathscr{A}, \tau)$, and set

$$x^N_{a,b} = \begin{cases} \dfrac{x_{a,b} + \mathbf{i}y_{a,b}}{\sqrt{2N}} & \text{if } a < b \\ \dfrac{x_{a,b}}{\sqrt{N}} & \text{if } a = b \\ \dfrac{x_{a,b} - \mathbf{i}y_{a,b}}{\sqrt{2N}} & \text{else.} \end{cases}$$

Then if we set $\tau_N := \tau \circ \mathrm{tr}_N$, one has that the trace of the k-th moment of x^N is equal to the right hand side of Equation (2.4). Thus x^N is a free semicircular variable. Let us now define x_i^N just like x^N but where each x_i^N is generated with freely independent semicircular systems. Next, we set $\mathscr{A}_N$ to be the W^*-subalgebra of $\mathbb{M}_N(\mathscr{A})$ generated by $\mathbb{M}_N(\mathbb{C})$ and $(x_i^N)_{1\leq i\leq d}$ and we endow $\mathscr{A}_N$ with τ_N. We then have the following lemma.

Lemma 2.13. *With $(\mathscr{A}_N, \tau_N)$ built as above, we have that $(x_i^N)_{1\leq i\leq d}$ is a free semicircular system, free from $\mathbb{M}_N(\mathbb{C})$.*

Thus one can view $(\mathscr{A}_N, \tau_N)$ as the free product of $\mathbb{M}_N(\mathbb{C})$ with the W^*-algebra generated by a free semicircular system. For a proof of Lemma 2.13, see [76, Proposition 2.3].

In particular, since the law of $(x_i^N)_{1\le i\le d}$ does not depend on N, we will often simply denote it $(x_i)_{1\le i\le d}$. Note that when restricted to $\mathbb{M}_N(\mathbb{C})$, τ_N is just the regular normalized trace on matrices. Similarly, the restriction of τ_N to the W^*-algebra generated by $(x_i^N)_{1\le i\le d}$ does not depends on N and hence will simply be denoted by τ.

2.2. Elliptic Brownian Motions, GL($N, \mathbb{C}$), and the Large-N Limit.

We summarize the constructions in Section 1.2 here, with further discussion of their large-N limits.

Definition 2.14. *Let $(X^N(t))_{t\ge0}$ and $(Y^N(t))_{t\ge0}$ be independent Hermitian Brownian motions of size N. Given $a, b, \theta \in \mathbb{R}$ such that $ab \neq 0$, we set*

$$\varrho = a^2 + b^2, \quad \zeta = e^{2\mathbf{i}\theta}(a^2 - b^2).$$

Given any $(\varrho, \zeta) \in \mathbb{R}_+ \times \mathbb{C}$ with $|\zeta| < \varrho$, there are a, b, θ satisfying these equations; cf. (1.9).

Then we define the general **elliptic Brownian motion** $W^N_{\varrho,\zeta}$ *by*

$$W^N_{\varrho,\zeta} = e^{\mathbf{i}\theta}\left(aX^N + \mathbf{i}bY^N\right).$$

Similarly, if $(x(t))_{t\ge0}$ and $(y(t))_{t\ge0}$ are freely independent free semicircular Brownian motion, the **free elliptic Brownian motion** $w_{\varrho,\zeta}$ *is defined as*

$$w_{\varrho,\zeta} = e^{\mathbf{i}\theta}\left(ax + \mathbf{i}by\right).$$

We will typically suppress the explicit (ϱ, ζ) dependence in the notation unless it is necessary for disambiguation within a given statement or proof, using the notations W^N and w instead of $W^N_{\varrho,\zeta}$ and $w_{\varrho,\zeta}$.

Of particular note is the case $(\varrho, \zeta) = (0, 1)$ (corresponding to $a = b = 1$ with θ undetermined); here W^N is a Ginibre Brownian motion.

Definition 2.15. *We define the (ϱ, ζ)-Brownian motion $(B^N_{\varrho,\zeta}(t))_{t\ge0}$ as the solution of the SDE (1.11), restated here:*

$$B^N_{\varrho,\zeta}(t) = I_N + \int_0^t B^N_{\varrho,\zeta}(s)\, dW^N_{\varrho,\zeta}(s) + \frac{\zeta}{2}\int_0^t B^N_{\varrho,\zeta}(s)\, ds.$$

We also define its free version $b_{\varrho,\zeta}$ by replacing the independent Hermitian Brownian motions in the SDE above by freely independent self-adjoint free Brownian motions.

Note that, in order to make it easier to read the equations, we will use the notations B^N and b instead of $B^N_{\varrho,\zeta}$ and $b_{\varrho,\zeta}$.

The "flat" Brownian motions $W^N = W^N_{\varrho,\zeta}$ are (by design) invariant under unitary conjugation; in other words, for a fixed $U \in \mathbf{U}(N)$, the process $\mathrm{Ad}_U W^N$ defined by

$$\mathrm{Ad}_U W^N(t) := U\, W^N(t)\, U^* = \int_0^t U\, dW^N(t)\, U^*$$

has the same law (as a process) as W^N. It follows that the same holds true for the process B^N.

Lemma 2.16. *If $U \in \mathrm{U}(N)$, the $\mathrm{GL}(N,\mathbb{C})$ Brownian motion $B^N = B^N_{\varrho,\zeta}$ is invariant in law under conjugation by U: the process $t \mapsto UB^N(t)U^*$ has the same law as B^N.*

Proof. Directly from (1.11) we have

$$\begin{aligned}\mathrm{Ad}_U B^N(t) &= UIU^* + U\int_0^t B^N(s)\,dW^N(s)\,U^* + \frac{\zeta}{2}U\int_0^t B^N(s)\,ds\,U^* \\ &= I + \int_0^t UB^N(s)U^*\,UdW^N(s)U^* + \frac{\zeta}{2}\int_0^t UB^N(s)U^*\,ds \\ &= I + \int_0^t UB^N(s)U^*\,d(\mathrm{Ad}_U W^N(s)) + \frac{\zeta}{2}\int_0^t UB^N(s)U^*\,ds.\end{aligned}$$

Thus UB^NU^* satisfies the same SDE, with the same initial distribution I, as B^N does with respect to $\mathrm{Ad}_U W^N \overset{\mathrm{d}}{=} W^N$; thus $UB^NU^* \overset{\mathrm{d}}{=} B^N$. □

The free multiplicative Brownian motion $b_{\varrho,\zeta}$ is the large-N limit (in finite dimensional $*$-distributions) of the $\mathrm{GL}(N,\mathbb{C})$ Brownian motion $B^N_{\varrho,\zeta}$; this was proved for $\zeta \in \mathbb{R}$ in [59] and generalized to complex ζ with $|\zeta| < \varrho$ (and strong convergence) in [11]. We need this convergence paired with an arbitrary initial condition, stated in Lemma 1.9; we now provide the proof.

Proof of Lemma 1.9: Fix $t \geq 0$. [24, 59, 11] proved that $B^N(t) \rightharpoonup b(t)$ in $*$-distribution; by assumption of the Lemma, $B_0^N \rightharpoonup b_0$ in $*$-distribution. Hence, by Proposition 2.7, $(B_0^N, U^N B^N(t) U^{N\,*})$ converge a.s. in $*$-distribution to $(b_0, ub(t)u^*)$.

Now, let $f \in C_b(\mathbb{M}_N(\mathbb{C}))$, and note that

$$\begin{aligned}&\mathbb{E}\left[f(B_0^N, U^N B^N(t) U^{N\,*})\right] \\ &= \int_{\mathrm{U}(N)} \mathrm{Haar}(dU) \int_{\mathbb{M}_N(\mathbb{C})} \mathrm{Law}_{B_0^N}(dB_0) \int_{\mathbb{M}_N(\mathbb{C})} \mathrm{Law}_{B^N(t)}(dB_1)\; f(B_0, UB_1U^*)\end{aligned}$$

where we have used the change-of-variables formula and the independence of B_0^N and $B^N(t)$.

In the inside integral, U is a fixed unitary matrix; hence, by Lemma 2.16, the law of $B^N(t)$ is invariant under conjugation by U. Thus

$$\int_{\mathbb{M}_N(\mathbb{C})} \mathrm{Law}_{B^N(t)}(dB_1)\; f(B_0, UB_1U^*) = \int_{\mathbb{M}_N(\mathbb{C})} \mathrm{Law}_{B^N(t)}(dB_1)\; f(B_0, B_1)$$

and reassembling this establishes that

$$\mathbb{E}\left[f(B_0^N, U^N B^N(t) U^{N\,*})\right] = \mathbb{E}\left[f(B_0^N, B^N(t))\right].$$

Since this holds for all $f \in C_b(\mathbb{M}_N(\mathbb{C}))$, it follows that $(B_0^N, U^N B^N(t) U^{N\,*}) \overset{\mathrm{d}}{=} (B_0^N, B^N(t))$. Thus

$$(B_0^N, B^N(t)) \overset{\mathrm{d}}{=} (B_0^N, U^N B^N(t) U^{N\,*}) \rightharpoonup (b_0, ub(t)u^*) \overset{\mathrm{d}}{=} (b_0, b(t)) \quad \text{a.s.}$$

where the last equality in $*$-distribution comes from the fact that $ub(t)u^*$ has the same $*$-distribution as $b(t)$, and the pair $b_0, b(t)$ is already freely independent. Thence, it follows that $B_0^N B^N(t)$ converges a.s. in $*$-distribution to $b_0 b(t)$, as claimed. □

As mentioned in the introduction, we will use a discretized version of the (ϱ,ζ)-Brownian motion which we define below.

Definition 2.17. *We define the discrete approximation of the (ϱ, ζ)-Brownian motion by*

$$B_n^N(t) = \prod_{i=1}^{\lfloor t2^n \rfloor} \left(I_N + W_{\varrho,\zeta}^N\left(\frac{i}{2^n}\right) - W_{\varrho,\zeta}^N\left(\frac{i-1}{2^n}\right) + \frac{\zeta}{2^{n+1}} I_N \right).$$

We also define its free version $b_n(t)$ by replacing $W_{\varrho,\zeta}^N$ by $w_{\varrho,\zeta}$.

2.3. **Schwinger–Dyson Equations.**

We will need the notion of noncommutative differentials and Gaussian integration by parts, also known as Schwinger-Dyson Equation.

Definition 2.18. *We define $\mathcal{P}_d = \mathbb{C}\langle X_1, \ldots, X_d \rangle$ the set of noncommutative polynomials in d variables.*

Definition 2.19. *If $1 \leq i \leq d$, one defines the noncommutative derivatives $\partial_i : \mathcal{P}_d \longrightarrow \mathcal{P}_d^{\otimes 2}$ on a monomial M by*

$$\partial_i M = \sum_{M = A X_i B} A \otimes B,$$

and then extend it by linearity to all polynomials. We then define the cyclic derivatives $D_i = m \circ \partial_i$ with $m : A \otimes B \mapsto BA$.

For a proof of the following proposition, see for example Proposition 2.10 of [27].

Proposition 2.20. *Let X^N be a family of d independent GUE random matrices of size N, $Q \in \mathcal{P}_d$, then for any i,*

$$\mathbb{E}\left[\mathrm{tr}_N(X_i^N Q(X^N))\right] = \mathbb{E}\left[\mathrm{tr}_N \otimes \mathrm{tr}_N\left(\partial_i Q(X^N)\right)\right],$$

where tr_N is the normalized trace.

Proposition 2.21. *Let x be a family of d free semicircular variables, $Q \in \mathcal{P}_d$, then for any i,*

$$\tau(x_i Q(x)) = \tau \otimes \tau\left(\partial_i Q(X^N)\right),$$

where τ is the trace on $\mathcal{C}^*(x)$.

One of the core element of the proof of most of the technical results in this paper is a specific asymptotic expansion obtained in [74]. In order to state it, we will need the following notations.

Definition 2.22. *Let us assume that we are given,*

- *X^N a d-tuple of independent GUE random matrices,*
- *$(x_i^m)_{i \in [1,d], m \in \mathbb{N}}$ free semicircular variables,*
- *$I = \{I_1, \ldots, I_{2n}\} \in (\mathbb{N} \setminus \{0\})^{2n}$ a collection of $2n$ integers,*
- *$T_n = \{t_1, \ldots, t_{2n}\}$ a sequence of non-negative real number.*

Then with $\widetilde{t}_0 = 0$, $T_n = \{\widetilde{t}_1, \ldots, \widetilde{t}_{2n}\}$ the same set but with elements sorted in increasing order, we set

$$X_{i,I}^{N,T_n} = \sum_{l=1}^{2n} (e^{-\widetilde{t}_{l-1}} - e^{-\widetilde{t}_l})^{1/2} x_i^{I_l} + e^{-\widetilde{t}_{2n}/2} X_i^N,$$

$$x_{i,I}^{T_n} = \sum_{l=1}^{2n} (e^{-\widetilde{t}_{l-1}} - e^{-\widetilde{t}_l})^{1/2} x_i^{I_l} + e^{-\widetilde{t}_{2n}/2} x_i^0.$$

Note that for any given I and T_n, the family $(x_{i,I}^{T_n})_{1\leq i\leq d}$ is a free semicircular system.

Definition 2.23. *Given $J_n \subset (\mathbb{N}\setminus\{0\})^{2n}$ a collection of sets who are themselves a family of $2n$ positive integers. Then one can consider non-commutative polynomials in $(X_{i,I})_{1\leq i\leq n, I\in J_n}$, and similarly to Definition 2.19, one can define $\partial_{i,I}$ the non-commutative differential with respect to $X_{i,I}$. In this context, we set*

$$\partial_i = \sum_{I\in J_n} \partial_{i,I}.$$

Finally, note that if there exists a bijection α between J_n and another collection of sets K_n, then given a polynomial P in $(X_{i,I})_{1\leq i\leq n, I\in J_n}$, one can view it as a polynomial in $(X_{i,K})_{1\leq i\leq n, K\in K_n}$ by defining

$$P\left((X_{i,K})_{1\leq i\leq n, K\in K_n}\right) := P\left((X_{i,\alpha(I)})_{1\leq i\leq n, I\in J_n}\right).$$

Definition 2.24. *For $n\geq 0$, let us assume that we are given $J_n \subset (\mathbb{N}\setminus\{0\})^{2n}$ (with convention $J_0 = \{\emptyset\}$). Let us also assume that we are given $J_{n+1}^{l,1}, J_{n+1}^{l,2}, \widetilde{J}_{n+1}^{l,1}, \widetilde{J}_{n+1}^{l,2} \subset (\mathbb{N}\setminus\{0\})^{2n+2}$ as well as a bijection between each of those sets and J_n. Hence, given A, B, C, D polynomials in $(X_{i,I})_{i\in[1,d], I\in J_n}$, one can define*

$$\begin{aligned}\Theta_l(A\otimes B\otimes C\otimes D) :=& B\left((X_{i,I})_{i\in[1,d], I\in J_{n+1}^{l,1}}\right) A\left((X_{i,I})_{i\in[1,d], I\in \widetilde{J}_{n+1}^{l,1}}\right)\\ & D\left(\left(\widetilde{X}_{i,I}\right)_{i\in[1,d], I\in \widetilde{J}_{n+1}^{l,2}}\right) C\left((X_{i,I})_{i\in[1,d], I\in J_{n+1}^{l,2}}\right).\end{aligned}$$

Next, we define the following operators for l from 1 to $2n+1$,

$$L_l^n(Q) := \frac{1}{2} \sum_{\substack{1\leq i,j\leq d\\ I,J\in J_n\\ \text{such that } I_l = J_l}} \Theta_l\circ(\partial_{j,I}\otimes\partial_{j,J})\circ(\partial_i D_i)\, Q,$$

Note that if $l = 2n+1$ then $I_l = J_l$ is always satisfied since elements of J_n are collections of $2n$ integers. Finally, if T_{n+1} is a set of $2n+2$ numbers, with $\widetilde{T_n} = \{\widetilde{t}_1, \dots, \widetilde{t}_{2n}\}$ the set which contains the first $2n$ elements of T_{n+1} sorted in increasing order, we set

$$L^{T_{n+1}}(Q) := \mathbb{1}_{[\widetilde{t}_{2n}, t_{2n+2}]}(t_{2n+1}) L_{2n+1}^n(Q) + \sum_{1\leq l\leq 2n} \mathbb{1}_{[\widetilde{t}_{l-1}, \widetilde{t}_l]}(t_{2n+1}) L_l^n(Q). \tag{2.5}$$

Then we set

$$J_{n+1} = \bigcup_{1\leq l\leq 2n+1} J_{n+1}^{l,1}\cup J_{n+1}^{l,2}\cup \widetilde{J}_{n+1}^{l,1}\cup \widetilde{J}_{n+1}^{l,2}.$$

Thus $L^{T_{n+1}}$ is an operator which takes a polynomial in $(X_{i,I})_{i\in[1,d], I\in J_n}$ in input and returns a polynomial in $(X_{i,I})_{i\in[1,d], I\in J_{n+1}}$.

We can finally state our asymptotic expansion. For the reader interested in the proof, it is a direct corollary of Lemma 3.6 and Proposition 3.7 of [74].

Lemma 2.25. *With the notations of Definitions 2.22, 2.23, and 2.24, one can find $J_n^{l,1}, J_n^{l,2}, \widetilde{J}_n^{l,1}, \widetilde{J}_n^{l,2} \subset (\mathbb{N}\setminus\{0\})^{2n}$ for any $n\geq 1$ such that if we define*

$$A_i = \{t_{2i}\geq t_{2i-2}\geq\cdots\geq t_2\geq 0\}\cap\{\forall s\in[1,i], t_{2s}\geq t_{2s-1}\geq 0\}\subset\mathbb{R}^{2i},$$

then for any for any $Q \in \mathcal{P}_d$, and $k \geq 1$,

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{tr}_N\left(Q(X^N)\right)\right] \\
&= \sum_{0\leq i<k} \frac{1}{N^{2i}} \int_{A_i} e^{-t_{2i}-\cdots-t_1} \tau_N\Big(\left(L^{T_i}\dots L^{T_1}\right)(Q)(x^{T_i})\Big)\, dt_1\dots dt_{2i} \\
&\quad + \frac{1}{N^{2k}} \int_{A_k} e^{-t_{2k}-\cdots-t_1} \mathbb{E}\left[\tau_N\Big(\left(L^{T_k}\dots L^{T_1}\right)(Q)(X^{N,T_k})\Big)\right]\, dt_1\dots dt_{2k}.
\end{aligned}$$

Note that we do not explicitly define the sets J_n. Indeed their construction is quite lengthy and will not be used in this paper. For the reader interested we refer to Section 2.3 of [74].

Before concluding this subsection, let us prove the following proposition that allows us to control moments of semicircular variables and GUE random matrices by the one of Gaussian random variables.

Proposition 2.26. *Given M a monomial of degree k, then*

$$0 \leq \tau\left(M(x^{T_n})\right) \leq \mathbb{E}[M(g)], \tag{2.6}$$

$$0 \leq \mathbb{E}\left[\tau_N\left(M(X^{N,T_n})\right)\right] \leq \mathbb{E}[M(g)], \tag{2.7}$$

where $g = (g_{i,I})_{i\in[1,d],I\in J_n}$ are centered Gaussian random variables of variance 1 such that $g_{i,I}$ and $g_{j,J}$ are independent if $i \neq j$ and equal otherwise.

In particular, given $c_M \in \mathbb{C}$ and $a_M \in \mathbb{R}^+$ such that $|c_M| \leq a_M$, then

$$\left|\mathbb{E}\left[\tau_N\left(\sum_{M \text{ monomial}} c_M M(X^{N,T_n})\right)\right]\right| \leq \mathbb{E}\left[\sum_{M \text{ monomial}} a_M M(g)\right], \tag{2.8}$$

and similarly for x^{T_n} instead of X^{N,T_n}.

Proof. Thanks to Proposition 2.21, assuming that $M = X_{i,I}R$, we have that

$$\tau\left(M(x^{T_n})\right) = e^{-\widetilde{t}_{2n}}\tau\otimes\tau\left(\partial_{i,0}R(x^{T_n})\right) + \sum_{l=1}^{2n}(e^{-\widetilde{t}_{l-1}} - e^{-\widetilde{t}_l})\tau\otimes\tau\left(\partial_{i,I_l}R(x^{T_n})\right),$$

where ∂_{i,I_l} is the noncommutative partial differential with respect to x_{i,I_l}. Thus by a straight-forward induction $\tau\left(M(x^{T_n})\right) \geq 0$.

Let us remark that the product of the s-th and r-th moment of a centered Gaussian random variable of variance 1 is always bounded it $s+r$-th moment. Thus for any monomials A and B, $\mathbb{E}[A(g)]\mathbb{E}[B(g)] \leq \mathbb{E}[A(g)B(g)]$. Hence if we assume that Equation (2.6) is satisfied for monomials of degree strictly smaller than k, then

$$\tau\left(M(x^{T_n})\right) \leq \left(e^{-\widetilde{t}_{2n}} + \sum_{l=1}^{2n}(e^{-\widetilde{t}_{l-1}} - e^{-\widetilde{t}_l})\right)\mathbb{E}\left[\frac{\partial R(g)}{\partial g_i}\right] = \mathbb{E}\left[\frac{\partial R(g)}{\partial g_i}\right] = \mathbb{E}\left[g_i R(g)\right],$$

where we used Gaussian integration by part in the last equality. Hence the proof of Equation (2.6) by induction.

Let us now prove Equation (2.7). Thanks to Lemma 2.25 (or more precisely Lemma 3.6 of [74]), we have that

$$\mathbb{E}\left[\mathrm{tr}_N\left(M(X^{N,T_n})\right)\right] = \sum_{i=0}^{k/4}\frac{c_i}{N^{2i}},$$

where the coefficients c_i are linear combination with non-negative coefficients of traces of monomials in x^{T_l} for $l \geq n$. In particular, thanks to the first part of the proof, those coefficients are non-negative. Thus

$$0 \leq \mathbb{E}\left[\mathrm{tr}_N\left(M(X^{N,T_n})\right)\right] = \sum_{i=0}^{k/4} \frac{c_i}{N^{2i}} \leq \sum_{i=0}^{k/4} c_i = \mathbb{E}\left[\tau\left(M(X^{1,T_n})\right)\right].$$

However, if we set $G = (G_{i,I}^{T_n})_{i\in[1,d],I\in J_n}$ a family of independent centered Gaussian random variables of variance 1, then by using the first equation,

$$0 \leq \mathbb{E}\left[\mathrm{tr}_N\left(M(X^{N,T_n})\right)\right] \leq \mathbb{E}\left[M(G)\right].$$

Hence the conclusion by using the fact that if $g_1, \ldots, g_l$ are centered Gaussian random variables of variance 1, not necessarily independent but such that for all i, j, $\mathrm{Cov}(g_i, g_j) \geq 0$, then

$$\mathbb{E}\left[g_1 \ldots g_l\right] \leq \mathbb{E}[g_1^l].$$

□

3. Proof of Theorem 1.7: Small Time Norm Concentration

The aim of this section is to prove Theorem 1.7. To do so we will need the following lemmas.

Lemma 3.1. *One has for any $k \in \mathbb{N}$ that there exists a constant $C_{k,t}$ such that*

$$\sup_{n,N\geq 1} \mathbb{E}\left[\mathrm{tr}_N\left(|B_n^N(t)|^{2k}\right)\right] \leq C_{k,t}.$$

Besides this upper bound still stands if one replaces $B_n^N(t)$ by $b_n(t)$.

Proof. There exists a family of independent GUE random matrices $(X_i^N, Y_i^N)_{i\geq 1}$ such that

$$B_n^N(t) = \prod_{i=1}^{\lfloor t2^n \rfloor} \left(I_N + \frac{e^{\mathbf{i}\theta}}{2^{n/2}}\left(aX_i^N + \mathbf{i}bY_i^N\right) + \frac{\zeta}{2^{n+1}} I_N\right).$$

Thus, thanks to Proposition 2.26,

$$\begin{aligned}\mathbb{E}\left[\mathrm{tr}_N\left(|B_n^N(t)|^{2k}\right)\right] &\leq \mathbb{E}\left[\left|\prod_{i=1}^{\lfloor t2^n \rfloor} \left(I_N + \frac{1}{2^{n/2}}\left(|a|x_i + |b|y_i\right) + \frac{|a^2-b^2|}{2^{n+1}} I_N\right)\right|^{2k}\right] \\ &= \mathbb{E}\left[\left|\prod_{i=1}^{\lfloor t2^n \rfloor} \left(I_N + \sqrt{\frac{a^2+b^2}{2^n}} g_i + \frac{|a^2-b^2|}{2^{n+1}} I_N\right)\right|^{2k}\right],\end{aligned}$$

where x_i, y_i, g_i are independent centered Gaussian random variables. Thus

$$\begin{aligned}\mathbb{E}\left[\mathrm{tr}_N\left(|B_n^N(t)|^{2k}\right)\right] &\leq \prod_{i=1}^{\lfloor t2^n \rfloor} \mathbb{E}\left[\left(I_N + \sqrt{\frac{a^2+b^2}{2^n}} g_i + \frac{|a^2-b^2|}{2^{n+1}} I_N\right)^{2k}\right] \\ &= \mathbb{E}\left[\left(I_N + \sqrt{\frac{a^2+b^2}{2^n}} g_1 + \frac{|a^2-b^2|}{2^{n+1}} I_N\right)^{2k}\right]^{\lfloor t2^n \rfloor}.\end{aligned}$$

However, one can write

$$\mathbb{E}\left[\left(I_N + \sqrt{\frac{a^2+b^2}{2^n}}g_1 + \frac{|a^2-b^2|}{2^{n+1}}I_N\right)^{2k}\right] = 1 + \sum_{i=1}^{k}\frac{c_{i,k}}{2^{in}},$$

where the coefficients $c_{i,k}$ do not depend on n. Thus

$$\mathbb{E}\left[\mathrm{tr}_N\left(|B_n^N(t)|^{2k}\right)\right] \leq e^{t\sum_{i=1}^{k}\frac{c_{i,k}}{2^{(i-1)n}}}.$$

Hence the conclusion.

□

Lemma 3.2. *For any $n \geq 1$, with $\|X\|_2 = \mathbb{E}[\mathrm{tr}_N(XX^*)]^{1/2}$, we have that*

$$\left\|B_n^N(t) - B_{n-1}^N(t)\right\|_2^2 \leq C_\varrho \frac{t+1}{2^n} e^{K_\varrho t}.$$

Similarly, with $\|X\|_2 = \mathbb{E}[\tau(XX^*)]^{1/2}$, the equation above also stands if one replaces $B_n^N(t)$ by $b_n(t)$.

Proof. Given $(X_i^N, Y_i^N)_{i\geq 1}$ a family of independent GUE random matrices, then with

$$\Delta_i^n := I_N + \frac{ae^{\mathbf{i}\theta}}{2^{n/2}}X_i^N + \mathbf{i}\frac{be^{\mathbf{i}\theta}}{2^{n/2}}Y_i^N + \frac{\zeta}{2^{n+1}}I_N,$$

we have that

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{tr}_N\left(B_n^N(t)(B_n^N(t))^*\right)\right] \\
&= \mathbb{E}\left[\mathrm{tr}_N\left(\prod_{i=2}^{\lfloor t2^n\rfloor}\Delta_i^n\left(\prod_{i=2}^{\lfloor t2^n\rfloor}\Delta_i^n\right)^*\right)\right]\left|1+\frac{\zeta}{2^{n+1}}\right|^2 \\
&\quad + \mathbb{E}\left[\mathrm{tr}_N\left(\frac{ae^{\mathbf{i}\theta}X_1^N + \mathbf{i}be^{\mathbf{i}\theta}Y_1^N}{2^{n/2}}\prod_{i=2}^{\lfloor t2^n\rfloor}\Delta_i^n\left(\prod_{i=2}^{\lfloor t2^n\rfloor}\Delta_i^n\right)^*\right)\right]\left(1+\frac{\zeta^*}{2^{n+1}}\right) \\
&\quad + \left(1+\frac{\zeta}{2^{n+1}}\right)\mathbb{E}\left[\mathrm{tr}_N\left(\prod_{i=2}^{\lfloor t2^n\rfloor}\Delta_i^n\left(\prod_{i=2}^{\lfloor t2^n\rfloor}\Delta_i^n\right)^*\frac{ae^{-\mathbf{i}\theta}X_1^N - \mathbf{i}be^{-\mathbf{i}\theta}Y_1^N}{2^{n/2}}\right)\right] \\
&\quad + \mathbb{E}\left[\mathrm{tr}_N\left(\frac{ae^{\mathbf{i}\theta}X_1^N + \mathbf{i}be^{\mathbf{i}\theta}Y_1^N}{2^{n/2}}\prod_{i=2}^{\lfloor t2^n\rfloor}\Delta_i^n\left(\prod_{i=2}^{\lfloor t2^n\rfloor}\Delta_i^n\right)^*\frac{ae^{-\mathbf{i}\theta}X_1^N - \mathbf{i}be^{-\mathbf{i}\theta}Y_1^N}{2^{n/2}}\right)\right] \\
&= \mathbb{E}\left[\mathrm{tr}_N\left(\prod_{i=2}^{\lfloor t2^n\rfloor}\Delta_i^n\left(\prod_{i=2}^{\lfloor t2^n\rfloor}\Delta_i^n\right)^*\right)\right]\left(\left|1+\frac{\zeta}{2^{n+1}}\right|^2 + \frac{\varrho}{2^n}\right) \\
&= \mathbb{E}\left[\mathrm{tr}_N\left(\prod_{i=2}^{\lfloor t2^n\rfloor}\Delta_i^n\left(\prod_{i=2}^{\lfloor t2^n\rfloor}\Delta_i^n\right)^*\right)\right]\left(1+\frac{\varrho+\mathrm{Re}\,\zeta}{2^n}+\frac{|\zeta|^2}{4^{n+1}}\right),
\end{aligned}$$

where we used Proposition 2.20 to go from the second to third equality. Consequently by induction, we get that

$$\mathbb{E}\left[\mathrm{tr}_N\left(B_n^N(t)(B_n^N(t))^*\right)\right] = \left(1+\frac{\varrho+\mathrm{Re}\,\zeta}{2^n}+\frac{|\zeta|^2}{4^{n+1}}\right)^{\lfloor t2^n\rfloor}. \tag{3.1}$$

Similarly, if we set

$$\Gamma_i^n := I_N + \frac{ae^{\mathbf{i}\theta}}{2^{n/2}}(X_{2i-1}^N + X_{2i}) + \mathbf{i}\frac{be^{\mathbf{i}\theta}}{2^{n/2}}(Y_{2i-1}^N + Y_{2i}) + \frac{\zeta}{2^n} I_N,$$

we get that

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{tr}_N\left(B_n^N(t)(B_{n-1}^N(t))^*\right)\right] \\
&= \mathbb{E}\left[\mathrm{tr}_N\left(\prod_{i=1}^{\lfloor t2^{n-1}\rfloor} \Delta_{2i-1}^n\Delta_{2i}^n \prod_{i=2\lfloor t2^{n-1}\rfloor+1}^{\lfloor t2^n\rfloor} \Delta_i^n \times \left(\prod_{i=1}^{\lfloor t2^{n-1}\rfloor}\Gamma_i^n\right)^*\right)\right] \\
&= \mathbb{E}\left[\mathrm{tr}_N\left(\prod_{i=1}^{\lfloor t2^{n-1}\rfloor} \Delta_{2i-1}^n\Delta_{2i}^n\left(\prod_{i=1}^{\lfloor t2^{n-1}\rfloor}\Gamma_i^n\right)^*\right)\right]\left(1+\frac{\zeta}{2^{n+1}}\right)^{\lfloor t2^n\rfloor - 2\lfloor t2^{n-1}\rfloor}
\end{aligned}$$

where we used the fact that

$$\prod_{i=2\lfloor t2^{n-1}\rfloor+1}^{\lfloor t2^n\rfloor} \Delta_i^n$$

is independent of other random matrices. Consequently, keeping in mind that

$$\begin{aligned}
\Delta_{2i-1}^n\Delta_{2i}^n = &\left(1+\frac{\zeta}{2^{n+1}}\right)\left(\frac{ae^{\mathbf{i}\theta}}{2^{n/2}}(X_{2i-1}^N + X_{2i}^N) + \mathbf{i}\frac{be^{\mathbf{i}\theta}}{2^{n/2}}(Y_{2i-1}^N + Y_{2i}^N)\right) \\
&+ \left(\frac{ae^{\mathbf{i}\theta}}{2^{n/2}}X_{2i-1}^N + \mathbf{i}\frac{be^{\mathbf{i}\theta}}{2^{n/2}}Y_{2i-1}^N\right)\left(\frac{ae^{\mathbf{i}\theta}}{2^{n/2}}X_{2i}^N + \mathbf{i}\frac{be^{\mathbf{i}\theta}}{2^{n/2}}Y_{2i}^N\right) \\
&+ \left(1+\frac{\zeta}{2^{n+1}}\right)^2 I_N,
\end{aligned}$$

by using once again Proposition 2.20, we get that

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{tr}_N\left(B_n^N(t)(B_{n-1}^N(t))^*\right)\right]\left(1+\frac{\zeta}{2^{n+1}}\right)^{-\lfloor t2^n\rfloor + 2\lfloor t2^{n-1}\rfloor} \\
&= \mathbb{E}\left[\mathrm{tr}_N\left(\prod_{i=2}^{\lfloor t2^{n-1}\rfloor} \Delta_{2i-1}^n\Delta_{2i}^n\left(\prod_{i=2}^{\lfloor t2^{n-1}\rfloor}\Gamma_i^n\right)^*\right)\right] \\
&\quad\times\left(\left(1+\frac{\zeta}{2^{n+1}}\right)^2\left(1+\frac{\overline{\zeta}}{2^n}\right) + \left(1+\frac{\zeta}{2^{n+1}}\right)\frac{\varrho}{2^{n-1}}\right) \\
&= \mathbb{E}\left[\mathrm{tr}_N\left(\prod_{i=2}^{\lfloor t2^{n-1}\rfloor} \Delta_{2i-1}^n\Delta_{2i}^n\left(\prod_{i=2}^{\lfloor t2^{n-1}\rfloor}\Gamma_i^n\right)^*\right)\right]\left(1+\frac{\varrho+\mathrm{Re}\,\zeta}{2^{n-1}} + \frac{C^{n,\varrho}}{4^n}\right),
\end{aligned}$$

where $C^{n,\varrho}$ is uniformly bounded by a constant that only depends on ϱ. Hence by induction, we get that

$$\begin{aligned}
\mathbb{E}\left[\mathrm{tr}_N\left(B_n^N(t)(B_{n-1}^N(t))^*\right)\right] = &\left(1+\frac{\varrho+\mathrm{Re}\,\zeta}{2^{n-1}} + \frac{C^{n,\varrho}}{4^n}\right)^{\lfloor t2^{n-1}\rfloor} \\
&\times\left(1+\frac{\zeta}{2^{n+1}}\right)^{\lfloor t2^n\rfloor - 2\lfloor t2^{n-1}\rfloor}.
\end{aligned}$$

Thus we have

$$\left\|B_n^N(t) - B_{n-1}^N(t)\right\|_2^2$$

$$
\begin{aligned}
&= \mathbb{E}\left[\mathrm{tr}_N\left(|B_n^N(t)|^2\right)\right] + \mathbb{E}\left[\mathrm{tr}_N\left(|B_{n-1}^N(t)|^2\right)\right] - 2\,\mathrm{Re}\,\mathbb{E}\left[\mathrm{tr}_N\left(B_n^N(t)(B_{n-1}^N(t))^*\right)\right] \\
&= \left(1 + \frac{\varrho + \mathrm{Re}\,\zeta}{2^n} + \frac{|\zeta|^2}{4^{n+1}}\right)^{\lfloor t2^n \rfloor} + \left(1 + \frac{\varrho + \mathrm{Re}\,\zeta}{2^{n-1}} + \frac{|\zeta|^2}{4^n}\right)^{\lfloor t2^{n-1} \rfloor} \\
&\quad - 2\,\mathrm{Re}\left(\left(1 + \frac{\varrho + \mathrm{Re}\,\zeta}{2^{n-1}} + \frac{C^{n,\varrho}}{4^n}\right)^{\lfloor t2^{n-1} \rfloor}\left(1 + \frac{\zeta}{2^{n+1}}\right)^{\lfloor t2^n \rfloor - 2\lfloor t2^{n-1} \rfloor}\right).
\end{aligned}
$$

Thus, given that $|\zeta| \leq |\varrho|$, one can find constants C_ϱ, K_ϱ which only depends on ϱ such that

$$
\left\|B_n^N(t) - B_{n-1}^N(t)\right\|_2^2 \leq C_\varrho \frac{t+1}{2^n} e^{K_\varrho t}.
$$

□

Proposition 3.3. *For any $T \geq 0$, $(B_n^N(t))_{t\geq 0}$ converges in $L^2(\Omega \times [0,T])$ towards a (ϱ,ζ)-Brownian motion $(B^N(t))_{t\geq 0}$. Besides for any $t \in \mathbb{R}$, there exists a constant $C_{\varrho,t}$ such that*

$$
\left\|B^N(t) - B_n^N(t)\right\|_2 \leq \frac{C_{\varrho,t}}{2^{n/2}}.
$$

Besides one has the same result when one replaces B_n^N and B^N by their free equivalent.

Proof. Thanks to Lemma 3.2, $(B_n^N)_{n\geq 0}$ is a Cauchy sequence in $L^2(\Omega \times [0,T])$, hence converges towards a stochastic process B^N. Besides, for any $t \in \mathbb{R}$,

$$
\left\|B^N(t) - B_n^N(t)\right\|_2 \leq \sum_{s \geq n+1} \sqrt{C_\varrho \frac{t+1}{2^s} e^{K_\varrho t}} \leq \frac{\sqrt{C_\varrho(t+1)} e^{K_\varrho t/2}}{2^{n/2}(\sqrt{2}-1)}. \tag{3.2}
$$

Let us now prove that B^N is a (ϱ,ζ)-Brownian motion, one has that

$$
\begin{aligned}
&\int_0^t B_n^N(s) dW^N(s) \\
&= \sum_{j=1}^{\lfloor t2^n \rfloor} \int_{\frac{j-1}{2^n}}^{\frac{j}{2^n}} B_n^N\left(\frac{j-1}{2^n}\right) dW^N(s) + \int_{\frac{\lfloor t2^n \rfloor}{2^n}}^t B_n^N\left(\frac{\lfloor t2^n \rfloor}{2^n}\right) dW^N(s) \\
&= \sum_{j=1}^{\lfloor t2^n \rfloor} B_n^N\left(\frac{j-1}{2^n}\right)\left(W^N\left(\frac{j}{2^n}\right) - W^N\left(\frac{j-1}{2^n}\right)\right) \\
&\qquad\qquad + B_n^N\left(\frac{\lfloor t2^n \rfloor}{2^n}\right)\left(W^N(t) - W^N\left(\frac{\lfloor t2^n \rfloor}{2^n}\right)\right) \\
&= \sum_{j=1}^{\lfloor t2^n \rfloor}\left(B_n^N\left(\frac{j}{2^n}\right) - B_n^N\left(\frac{j-1}{2^n}\right)\right) - \frac{\zeta}{2^{n+1}} \sum_{j=1}^{\lfloor t2^n \rfloor} B_n^N\left(\frac{j-1}{2^n}\right) \\
&\qquad\qquad + B_n^N\left(\frac{\lfloor t2^n \rfloor}{2^n}\right)\left(W^N(t) - W^N\left(\frac{\lfloor t2^n \rfloor}{2^n}\right)\right) \\
&= B_n^N\left(\frac{\lfloor t2^n \rfloor}{2^n}\right) - I_N - \frac{\zeta}{2}\int_0^{\lfloor t2^n \rfloor/2^n} B_n^N(s) ds \\
&\qquad\qquad + B_n^N\left(\frac{\lfloor t2^n \rfloor}{2^n}\right)\left(W^N(t) - W^N\left(\frac{\lfloor t2^n \rfloor}{2^n}\right)\right)
\end{aligned}
$$

$$= B_n^N(t) - I_N - \frac{\zeta}{2}\int_0^t B_n^N(s)ds + B_n^N\left(\frac{\lfloor t2^n\rfloor}{2^n}\right)\left(W^N(t) - W^N\left(\frac{\lfloor t2^n\rfloor}{2^n}\right) + \left(t - \frac{\lfloor t2^n\rfloor}{2^n}\right)I_N\right).$$

Thanks to Lemma 3.1, we have that

$$\begin{aligned}&\left\|B_n^N(t) - I_N - \int_0^t B_n^N(s)dW^N(s) - \frac{\zeta}{2}\int_0^t B_n^N(s)ds\right\|_2^2 \\ &= \mathcal{O}\left(\left\|W^N(t) - W^N\left(\frac{\lfloor t2^n\rfloor}{2^n}\right) + \left(t - \frac{\lfloor t2^n\rfloor}{2^n}\right)I_N\right\|_2^2\right) \\ &= \mathcal{O}(2^{-n}).\end{aligned} \tag{3.3}$$

Besides

$$\begin{aligned}&\left\|\int_0^t B_n^N(s)dW^N(s) - \int_0^t B^N(s)dW^N(s)\right\|_2^2 \\ &= \left\|\int_0^t (B_n^N(s) - B^N(s))dW^N(s)\right\|_2^2 \\ &= \mathbb{E}\left[\mathrm{tr}_N\left(\int_0^t (B_n^N(s) - B^N(s))dW^N(s)\left(\int_0^t (B_n^N(s) - B^N(s))dW^N(s)\right)^*\right)\right] \\ &= \mathbb{E}\Bigg[\sum_{1\le i,j,l,a,b,c\le N}\mathrm{tr}_N\Bigg(\int_0^t (B_n^N(s) - B^N(s))_{i,j}d(W^N(s))_{j,l}E_{i,l} \\ &\qquad\qquad \times\left(\int_0^t (B_n^N(s) - B^N(s))_{a,b}d(W^N(s))_{b,c}E_{a,c}\right)^*\Bigg)\Bigg] \\ &= \frac{1}{N}\sum_{1\le i,j,l,b\le N}\mathbb{E}\Bigg[\int_0^t (B_n^N(s) - B^N(s))_{i,j}d(W^N(s))_{j,l} \\ &\qquad\qquad \times\int_0^t \overline{(B_n^N(s) - B^N(s))_{i,b}}\ \overline{d(W^N(s))_{b,l}}\Bigg] \\ &= \frac{2\varrho}{N^2}\sum_{1\le i,j,l\le N}\mathbb{E}\left[\int_0^t \left|(B_n^N(s) - B^N(s))_{i,j}\right|^2 ds\right] \\ &= 2\varrho\int_0^t \left\|B_n^N(s) - B^N(s)\right\|_2^2 ds \\ &\le \frac{\int_0^t C_\varrho(s+1)e^{K_\varrho s}ds}{2^n(\sqrt{2}-1)^2},\end{aligned}$$

where we used Equation (3.2) in the last line. Thus in combination with Equation (3.3), as well as the fact that

$$\left\|\int_0^t (B_n^N(s) - B^N(s))ds\right\|_2 \le \int_0^t \left\|B_n^N(s) - B^N(s)\right\|_2 ds.$$

we get that

$$B^N(t) = I_N + \int_0^t B^N(s)dW^N(s) + \frac{\zeta}{2}\int_0^t B^N(s)ds.$$

Hence B^N is a (ϱ, ζ)-Brownian motion.

□

Proposition 3.4. *Given $(b(t))_{t\geq 0}$ a (ϱ, ζ)-free Brownian motion driven by $(w(t))_{t\geq 0}$, there exists constants C_ϱ, K_ϱ, such that*

$$\|b(t)\| \leq C_\varrho e^{K_\varrho t}, \quad \|b(t) - 1\| \leq \sqrt{t}\ C_\varrho e^{K_\varrho t}, \quad \|b(t) - 1 - w(t)\| \leq t\ C_\varrho e^{K_\varrho t}.$$

Proof. Note that

$$b(t) = 1 + \int_0^t b(s)\ dw(s) + \frac{\zeta}{2}\int_0^t b(s)ds,$$
$$b(t) - 1 = \int_0^t b(s)\ dw(s) + \frac{\zeta}{2}\int_0^t b(s)ds,$$
$$b(t) - 1 - w(t) = \int_0^t (b(s) - 1)\ dw(s) + \frac{\zeta}{2}\int_0^t b(s)ds.$$

Besides $w = e^{\mathbf{i}\theta}(ax + \mathbf{i}by)$ where x, y are usual self-adjoint free Brownian motions. Thus by using the Burkholder-Gundy inequality for free stochastic process, see Theorem 3.2.1 of [19], coupled with the fact that

$$\left\|\int_0^t b(s)ds\right\| \leq \left(\int_0^t \|b(s)\|^2 ds\right)^{1/2},$$

and $|\zeta| \leq \varrho$, we get that

$$\|b(t)\| \leq 1 + \left(4\sqrt{2\varrho} + \frac{\varrho}{2}\right)\left(\int_0^t \|b(s)\|^2 ds\right)^{1/2},$$
$$\|b(t) - 1\| \leq \left(4\sqrt{2\varrho} + \frac{\varrho}{2}\right)\left(\int_0^t \|b(s)\|^2 ds\right)^{1/2},$$
$$\|b(t) - 1 - w(t)\| \leq \sqrt{32\varrho}\left(\sqrt{32\varrho} + \frac{\varrho}{2}\right)\left(\int_0^t\int_0^s \|b(u)\|^2 duds\right)^{1/2} + \frac{\varrho}{2}\int_0^t \|b(s)\|\, ds.$$

By squaring the first equation we get that

$$\|b(t)\|^2 \leq 2 + \left(4\sqrt{2\varrho} + \frac{\varrho}{2}\right)^2 \int_0^t \|b(s)\|^2 ds,$$

and by Gronwall's inequality,

$$\|b(t)\|^2 \leq 2e^{\left(4\sqrt{2\varrho}+\frac{\varrho}{2}\right)^2 t}.$$

Consequently, we have that

$$\|b(t) - 1\| \leq \left(4\sqrt{2\varrho} + \frac{\varrho}{2}\right)\left(\int_0^t 2e^{\left(4\sqrt{2\varrho}+\frac{\varrho}{2}\right)^2 s} ds\right)^{1/2}$$
$$\leq \left(4\sqrt{2\varrho} + \frac{\varrho}{2}\right)\sqrt{2t}\ e^{\left(4\sqrt{2\varrho}+\frac{\varrho}{2}\right)^2 t/2},$$

$$\|b(t) - 1 - w(t)\| \leq 4\sqrt{2\varrho}\left(4\sqrt{2\varrho} + \frac{\varrho}{2}\right)\left(\int_0^t\int_0^s 2e^{\left(4\sqrt{2\varrho}+\frac{\varrho}{2}\right)^2 u} duds\right)^{1/2}$$

$$+\frac{\varrho}{2}\int_0^t \sqrt{2}e^{\left(4\sqrt{2\varrho}+\frac{\varrho}{2}\right)^2 s/2}ds$$
$$\leq \left(8\sqrt{\varrho}\left(4\sqrt{2\varrho}+\frac{\varrho}{2}\right)e^{\left(4\sqrt{2\varrho}+\frac{\varrho}{2}\right)^2 t/2}+\frac{\varrho}{\sqrt{2}}e^{\left(4\sqrt{2\varrho}+\frac{\varrho}{2}\right)^2 t/2}\right)t.$$

Hence the conclusion.

□

Theorem 3.5. *Given a (ϱ,ζ)-Brownian motion B^N, driven by W^N, there exists a constant $C_{\varrho,T}$ such that for all $k \geq 0$ and $t \leq T$,*

$$\mathbb{E}\left[\mathrm{tr}_N\left(\left|B^N(t)-I_N-W^N(t)\right|^{2k}\right)\right] \leq (tC_{\varrho,T})^{2k}e^{C_{\varrho,T}\frac{k^4}{N^2}}.$$

Proof. To begin with, by Cauchy-Schwarz and Lemma 3.1,

$$\left|\mathbb{E}\left[\mathrm{tr}_N\left(|B_n^N(t)|^{2k}\right)\right]-\mathbb{E}\left[\mathrm{tr}_N\left(|B^N(t)|^{2k}\right)\right]\right| \leq \mathcal{O}\left(\left\|B^N(t)-B_n^N(t)\right\|_{L^2}\right).$$

Consequently, thanks to Proposition 3.3, one has that

$$\mathbb{E}\left[\mathrm{tr}_N\left(\left|B^N(t)-I_N-W^N(t)\right|^{2k}\right)\right]$$
$$=\lim_{n\to\infty}\mathbb{E}\left[\mathrm{tr}_N\left(\left|B_n^N(t)-I_N-W^N\left(\frac{\lfloor 2^n t\rfloor}{2^n}\right)\right|^{2k}\right)\right].$$

Then we set

$$X^N=\left(\sqrt{2^n}\left(X^N_{\frac{i}{2^n}}-X^N_{\frac{i-1}{2^n}}\right)\right),\quad Y^N=\left(\sqrt{2^n}\left(Y^N_{\frac{i}{2^n}}-Y^N_{\frac{i-1}{2^n}}\right)\right)_{1\leq i\leq\frac{\lfloor 2^n t\rfloor}{2^n}},$$

where $(X_t^N)_{t\geq 0}$ and $(Y_t^N)_{t\geq 0}$ are the Hermitian Brownian motions used to define W^N in Definition 2.14. We also set Q_n a polynomial in X^N such that

$$Q_n(X^N,Y^N)=\left|B_n^N(t)-I_N-W^N\left(\frac{\lfloor 2^n t\rfloor}{2^n}\right)\right|^{2k}.$$

Consequently, with Lemma 2.25, we get that

$$\mathbb{E}\left[\mathrm{tr}_N\left(Q_n(X^N,Y^N)\right)\right] \tag{3.4}$$
$$=\sum_{0\leq i<s}\frac{1}{N^{2i}}\int_{A_i}e^{-t_{2i}-\cdots-t_1}\tau\Big(\left(L^{T_i}\dots L^{T_1}\right)(Q_n)(x^{T_i},y^{T_i})\Big)\,dt_1\dots dt_{2i}$$
$$+\frac{1}{N^{2s}}\int_{A_s}e^{-t_{2s}-\cdots-t_1}\mathbb{E}\left[\tau_N\Big(\left(L^{T_s}\dots L^{T_1}\right)(Q_n)(X^{N,T_s},Y^{N,T_s})\Big)\right]\,dt_1\dots dt_{2s}.$$

From there on, we will divide the proof in two steps where we upper bond each of the terms in the equation above.

Step 1: Let us upper bound the second term in the right hand side of Equation (3.4). First, let us remark that if we set P_n such that

$$P_n(X^N,Y^N):=\left|\prod_{1\leq s\leq\lfloor 2^n t\rfloor}\left(I_N\left(1+\frac{|\zeta|}{2^{n+1}}\right)+\frac{|a|X_s^N+|b|Y_s^N}{2^{n/2}}\right)\right|^{2k},$$

then if we write Q_n and P_n as linear combinations of monomials M, with coefficients c_M and a_M respectively, then $a_M\in\mathbb{R}^+$ and $|c_M|\leq a_M$. Note that those properties of the polynomials Q_n and P_n are conserved by applying the operator L^{T_i}. Thus,

with $g = (g_i)_{i\in[1,d],I\in J_n}$ where $(g_i)_i$ are independent centered Gaussian random variable of variance 1, $\widetilde{g}$ an independent copy of g, thanks to Proposition 2.26,

$$\int_{A_s} e^{-t_{2s}-\cdots-t_1}\mathbb{E}\left[\tau_N\Big(\left(L^{T_s}\dots L^{T_1}\right)(Q_n)(X^{N,T_s})\Big)\right]\,dt_1\dots dt_{2s}$$
$$\leq \int_{A_s} e^{-t_{2s}-\cdots-t_1}dt_1\dots dt_{2s}\mathbb{E}\left[\left(L^{T_s}\dots L^{T_1}\right)(P_n)(g,\widetilde{g})\right].$$

Besides, for any $T_s \in A_s$, we get that $\left(L^{T_s}\dots L^{T_1}\right)(P_n)(g,\widetilde{g})$ is a linear combination of at most $(2k)^{4s}(\lfloor 2^n t\rfloor)^{2s}$ terms of the form

$$\left(\frac{\lambda}{2^n}\right)^{2s}\prod_{i=1}^{\lfloor 2^n t\rfloor}\left|1+\frac{|\zeta|}{2^{n+1}}+\frac{|a|g_i+|b|\widetilde{g}_i}{2^{n/2}}\right|^{2(k-r_i)},$$

where $r_1+\dots+r_{\lfloor 2^n t\rfloor}=2s$. Consequently, for some constant $C_{k,t}$,

$$\mathbb{E}\left[\left(L^{T_s}\dots L^{T_1}\right)(P_n)(g,\widetilde{g})\right] \leq (2k)^{4s}\left(\varrho\frac{\lfloor 2^n t\rfloor}{2^n}\right)^{2s}\mathbb{E}\left[\left|1+\frac{g_i}{2^{n/2}}\right|^{2k}\right]^{\lfloor 2^n t\rfloor}$$
$$\leq (2k)^{4s}(\varrho t)^{2s}\,C_{k,t},$$

where we use the same argument as in the proof of Lemma 3.1 in the last line. And since,

$$\int_{A_s} e^{-t_{2s}-\cdots-t_1}dt_1\dots dt_{2s}\leq\frac{1}{s!},$$

one has that

$$\limsup_{n\to\infty}\int_{A_s} e^{-t_{2s}-\cdots-t_1}\mathbb{E}\left[\tau_N\Big(\left(L^{T_s}\dots L^{T_1}\right)(Q_n)(X^{N,T_n},Y^{N,T_n})\Big)\right]\,dt_1\dots dt_{2s}$$
$$\leq\frac{(2k)^{4s}(\varrho t)^{2s}}{s!}\,C_{k,t}. \tag{3.5}$$

Step 2: Let us now upper bound the first term in the right hand side of Equation (3.4). To do so we only need to upper bound the following term,

$$\limsup_{n\to\infty}\left|\int_{A_i} e^{-t_{2i}-\cdots-t_1}\tau\Big(\left(L^{T_i}\dots L^{T_1}\right)(Q_n)(x^{T_i},y^{T_i})\Big)\,dt_1\dots dt_{2i}\right|.$$

Note that one can write

$$Q_n(X,Y)=\prod_{1\leq i\leq\lfloor 2^n t\rfloor}\left(1+\frac{\zeta}{2^{n+1}}+e^{\mathbf{i}\theta}\frac{aX_i+\mathbf{i}bY_i}{2^{n/2}}\right)-\sum_{1\leq i\leq\lfloor 2^n t\rfloor}e^{\mathbf{i}\theta}\frac{aX_i+\mathbf{i}bY_i}{2^{n/2}}-1,$$

which means that its differential is the sum of two terms,

$$\partial_{X_j}Q_n(X,Y)$$
$$=\frac{ae^{\mathbf{i}\theta}}{2^{n/2}}\Bigg(\prod_{1\leq i<j}\left(1+\frac{\zeta}{2^{n+1}}+e^{\mathbf{i}\theta}\frac{aX_i+\mathbf{i}bY_i}{2^{n/2}}\right)$$
$$\otimes\prod_{j<i\leq\lfloor 2^n t\rfloor}\left(1+\frac{\zeta}{2^{n+1}}+e^{\mathbf{i}\theta}\frac{aX_i+\mathbf{i}bY_i}{2^{n/2}}\right)-1\otimes 1\Bigg)$$
$$=\frac{ae^{\mathbf{i}\theta}}{2^{n/2}}\left(\left(\prod_{1\leq i<j}\left(1+\frac{\zeta}{2^{n+1}}+e^{\mathbf{i}\theta}\frac{aX_i+\mathbf{i}bY_i}{2^{n/2}}\right)-1\right)\right.$$

$$\otimes \prod_{j<i\leq\lfloor 2^n t\rfloor}\left(1+\frac{\zeta}{2^{n+1}}+e^{\mathbf{i}\theta}\frac{aX_i+\mathbf{i}bY_i}{2^{n/2}}\right)$$
$$+1\otimes\left(\prod_{j<i\leq\lfloor 2^n t\rfloor}\left(1+\frac{\zeta}{2^{n+1}}+e^{\mathbf{i}\theta}\frac{aX_i+\mathbf{i}bY_i}{2^{n/2}}\right)-1\right)\Bigg).$$

And finally after differentiating twice, we get that

$$\begin{aligned}&(1\otimes\partial_{X_k})\circ\partial_{X_j}Q_n(X,Y)\\&=\frac{e^{2\mathbf{i}\theta}a^2}{2^n}\prod_{1\leq i<j}\left(1+\frac{\zeta}{2^{n+1}}+e^{\mathbf{i}\theta}\frac{aX_i+\mathbf{i}bY_i}{2^{n/2}}\right)\otimes\prod_{j<i<k}\left(1+\frac{\zeta}{2^{n+1}}+e^{\mathbf{i}\theta}\frac{aX_i+\mathbf{i}bY_i}{2^{n/2}}\right)\\&\quad\otimes\prod_{k<i\leq\lfloor 2^n t\rfloor}\left(1+\frac{\zeta}{2^{n+1}}+e^{\mathbf{i}\theta}\frac{aX_i+\mathbf{i}bY_i}{2^{n/2}}\right).\end{aligned}$$

One also have similar formulas for ∂_{Y_j}. Consequently, one has that $\left(L^{T_i}\dots L^{T_1}\right)(Q_n)$ is a linear combination of at most $2^{4i}\times(2k)^{4i}(\lfloor 2^n t\rfloor)^{2i}$ terms which are themselves products of $2k+4i$ terms of the following form (or of its adjoints),

$$\prod_{1\leq i\leq\lfloor 2^n t\rfloor}\left(1+\frac{\zeta}{2^{n+1}}+e^{\mathbf{i}\theta}\frac{aX_i+\mathbf{i}bY_i}{2^{n/2}}\right)-\sum_{1\leq i\leq\lfloor 2^n t\rfloor}e^{\mathbf{i}\theta}\frac{aX_i+\mathbf{i}bY_i}{2^{n/2}}-1,$$
$$\prod_{a\leq i\leq b}\left(1+\frac{\zeta}{2^{n+1}}+e^{\mathbf{i}\theta}\frac{aX_i+\mathbf{i}bY_i}{2^{n/2}}\right)-1,\ \text{or}\ \prod_{a\leq i\leq b}\left(1+\frac{\zeta}{2^{n+1}}+e^{\mathbf{i}\theta}\frac{aX_i+\mathbf{i}bY_i}{2^{n/2}}\right),$$

each evaluated in a given $(x_{i,J}^{T_i})_{1\leq i\leq d}$. Besides the coefficient in the linear combination of $\left(L^{T_i}\dots L^{T_1}\right)(Q_n)$ can be upper bounded by $\varrho^{2i}2^{-2in}$. Finally, with r_1 the number of products of the first type, r_2 the number of products of the second type, then necessarily $2r_1+r_2\geq 4k-4i$. Let us now remark that thanks to Propositions 3.3 and 3.4, we have with $(x_i,y_i)_{1\leq i\leq\lfloor 2^n t\rfloor}$ free circular variables that for all $m\in\mathbb{N}$, $1\leq a\leq b\leq\lfloor 2^n t\rfloor$,

$$\begin{aligned}&\left\|\prod_{1\leq i\leq\lfloor 2^n t\rfloor}\left(1+\frac{\zeta}{2^{n+1}}+e^{\mathbf{i}\theta}\frac{ax_i+\mathbf{i}by_i}{2^{n/2}}\right)-\sum_{1\leq i\leq\lfloor 2^n t\rfloor}e^{\mathbf{i}\theta}\frac{ax_i+\mathbf{i}by_i}{2^{n/2}}-1\right\|_m\\&\leq tC_\varrho e^{K_\varrho t}+\mathcal{O}(2^{-n/2}),\\&\left\|\prod_{a\leq i\leq b}\left(1+\frac{\zeta}{2^{n+1}}+e^{\mathbf{i}\theta}\frac{ax_i+\mathbf{i}by_i}{2^{n/2}}\right)-1\right\|_m\leq\sqrt{t}C_\varrho e^{K_\varrho t}+\mathcal{O}(2^{-n/2}),\\&\left\|\prod_{a\leq i\leq b}\left(1+\frac{\zeta}{2^{n+1}}+e^{\mathbf{i}\theta}\frac{ax_i+\mathbf{i}by_i}{2^{n/2}}\right)\right\|_m\leq C_\varrho e^{K_\varrho t}+\mathcal{O}(2^{-n/2}).\end{aligned}$$

Consequently, $\left(L^{T_i}\dots L^{T_1}\right)(Q_n)$ is a linear combination of at most $2^{4i}\times(2k)^{4i}(\lfloor 2^n t\rfloor)^{2i}$ terms which can be upper bounded thanks to the noncommutative Hölder inequality by

$$\frac{\varrho^{2i}}{2^{2ni}}t^{r_1+\frac{r_2}{2}}\left(C_\varrho e^{K_\varrho t}\right)^{2k+4i}+\mathcal{O}\left(2^{-\left(2i+\frac{1}{2}\right)n}\right).$$

One can upper bound $t^{r_1+\frac{r_2}{2}}$ by t^{k-2i} if $t\leq 1$ and otherwise we simply upper bound it by 1. Finally, by combining every previous estimates, we get that for

some constant $C_{\varrho,t}$,

$$\begin{aligned}
&\limsup_{n\to\infty}\left|\int_{A_i} e^{-t_{2i}-\cdots-t_1}\tau\Big(\left(L^{T_i}\dots L^{T_1}\right)(Q_n)(x^{T_i},y^{T_i})\Big)\,dt_1\dots dt_{2i}\right| \\
&\leq \limsup_{n\to\infty}\int_{A_i} e^{-t_{2i}-\cdots-t_1}dt_1\dots dt_{2i}\times k^{4i}(\lfloor 2^n t\rfloor)^{2i}\times\frac{1}{2^{2ni}}t^{2k-2i}(C_{\varrho,t})^{2k+i} \\
&\leq \frac{k^{4i}}{i!}t^{2k}(C_{\varrho,t})^{2k+i}.
\end{aligned}$$

Thus in combination with Equation (3.4) and (3.5), we get that for any s and k,

$$\begin{aligned}
\mathbb{E}\left[\mathrm{tr}_N\left(\left|B^N(t)-I_N-W^N(t)\right|^{2k}\right)\right] &\leq \sum_{i=0}^{s-1}\frac{k^{4i}(C_{\varrho,t})^i}{N^{2i}i!}(tC_{\varrho,t})^{2k}+\frac{(2k)^{4s}(\varrho t)^{2s}}{s!}\,C_{k,t}, \\
&\leq (tC_{\varrho,t})^{2k}e^{C_{\varrho,t}\frac{k^4}{N^2}}+\frac{(2k)^{4s}(\varrho t)^{2s}}{s!}\,C_{k,t}.
\end{aligned}$$

Hence the conclusion by letting s go to infinity.

□

Now we use Theorem 3.5 to conclude the proof of Theorem 1.7. First, from Theorem 3.5, one has that

$$\begin{aligned}
\mathbb{P}\left(\left\|B^N(t)-I_N-W^N(t)\right\|\geq\delta\right) &= \mathbb{P}\left(\left\|\left|B^N(t)-I_N-W^N(t)\right|^{2k}\right\|\geq\delta^{2k}\right) \\
&\leq \mathbb{P}\left(\mathrm{Tr}_N\left(\left|B^N(t)-I_N-W^N(t)\right|^{2k}\right)\geq\delta^{2k}\right) \\
&\leq N\frac{\mathbb{E}\left[\mathrm{tr}_N\left(\left|B^N(t)-I_N-W^N(t)\right|^{2k}\right)\right]}{\delta^{2k}} \\
&\leq N\left(\frac{tC_{\varrho,T}}{\delta}\right)^{2k}e^{C_{\varrho,T}\frac{k^4}{N^2}}.
\end{aligned}$$

Thus with $k_N=\lceil N^{2/3}\rceil$, and

$$c_{\varrho,T}=\sup_{N\geq 1}N^{\frac{1}{2k_N}}C_{\varrho,T}e^{C_{\varrho,T}\frac{k_N^3}{2N^2}},$$

we have that

$$\mathbb{P}\left(\left\|B^N(t)-I_N-W^N(t)\right\|\geq\delta\right)\leq\left(c_{\varrho,T}\times\frac{t}{\delta}\right)^{2k_N}.$$

Hence the conclusion.

We will also need a concentration estimate on the norm of $B^N(t)$. One could obtain it by using Theorem 3.5 coupled with an estimate on the norm of a GUE random matrix and the triangle inequality. However, following the strategy developed in the proof of Theorem 3.5 yields the desired bound with no need for another reference. Hence, we have the following result.

Theorem 3.6. *There exists a constant $c_{\varrho,T}$ such that for any $t\in[0,T]$ and $K>0$,*

$$\mathbb{P}\left(\left\|B^N(t)\right\|\geq K\right)\leq\left(\frac{c_{\varrho,T}}{K}\right)^{N^{2/3}}.$$

Proof. One has

$$\begin{aligned}\mathbb{P}\left(\left\|B^N(t)\right\| \geq K\right) &= \mathbb{P}\left(\left\|\left|B^N(t)\right|^{2k}\right\| \geq K^{2k}\right) \\ &\leq \mathbb{P}\left(\mathrm{Tr}_N\left(\left|B^N(t)\right|^{2k}\right) \geq K^{2k}\right) \\ &\leq \frac{N}{K^{2k}}\mathbb{E}\left[\mathrm{tr}_N\left(\left|B^N(t)\right|^{2k}\right)\right].\end{aligned}$$

The rest of the proof then follows exactly like those of Theorem 3.5 and 1.7. □

Letting $K > c_{\varrho,t}$ so that $r = c_{\varrho,t}/K \in (0,1)$, and noting that the decaying sequence $r^{N^{2/3}}$ is certainly summable for any $r \in (0,1)$, an application of the Borel–Cantelli lemma yields the following.

Corollary 3.7. *There is some constant $K = K_{\varrho,t} < \infty$ so that*

$$\mathbb{P}\{\|B^N(t)\| \leq K \text{ for all large } N\} = 1.$$

4. Estimates on Small Singular Values

In this section we prove the following:

Proposition 4.1. *Let $(B^N(t))_{t\geq 0}$ be a (ϱ,ζ)-Brownian motion with $|\zeta| < \varrho$. Let B_0^N, Z^N be $N\times N$ matrices, deterministic or random and independent of $(B^N(t))_{t\geq 0}$. Then the following hold for any $0 < t \leq T < \infty$.*

(a) For any $\delta, \alpha, \kappa > 0$ and all $N \geq (4/\alpha)^{3/2}$,

$$\mathbb{P}\big(\sigma_N(B_0^N B^N(t) + Z^N) \leq \delta,\ \sigma_N(B_0^N) \geq \kappa\big) \leq CN^2\delta^{1-\alpha}. \tag{4.1}$$

for some $C = C(\alpha,\kappa,\varrho,\zeta,T) < \infty$.

(b) For any $\kappa > 0$ and $0 \leq k \leq N-1$, except with probability $O(N^2e^{-k}) + \exp(-N^{2/3})$, on the event that $\sigma_N(B_0^N) \geq \kappa$, we have

$$\sigma_{N-\ell}(B_0^N B^N(t) + Z^N) \geq c(\ell/N)^2 \qquad \forall\ k \leq \ell \leq N-1 \tag{4.2}$$

for some $c = c(\kappa,\varrho,\zeta,T) > 0$.

Remark 4.2. *By a slight modification of the proof for part (a) we also obtain the bound*

$$\mathbb{P}\big(\sigma_N(B_0^N B^N(t) + Z^N) \leq \delta,\ \sigma_N(B_0^N) \geq \kappa\big) \leq O_{\alpha,\kappa,\varrho,\zeta,T}(N^2\delta) + \exp(-N^{2/3})$$

which is better for a certain range of δ.

To lighten notation we drop the superscript N from matrices $B^N(t), Z^N$, etc. for the remainder of this section.

The key to the proof of both parts of Proposition 4.1 is Theorem 1.7. We also need two lemmas. We have the following estimates for Ginibre matrices:

Lemma 4.3. *Let G be an $N\times N$ complex Ginibre matrix (having the distribution of $Z^N(1)$ in Definition 2.11), and let A be an arbitrary $N\times N$ matrix, deterministic or random and independent of G.*

(a) For any $\delta \geq 0$,

$$\mathbb{P}(\sigma_N(G+A) \leq \delta) \leq N^2\delta^2\,.$$

(b) For any $0 \leq k \leq N-1$,

$$\mathbb{P}(\sigma_{N-k}(G+A) \leq ck/N) \leq CNe^{-k}$$

where $C, c > 0$ are universal constants.

Proof. For (a) see [10, Lemma 3.3]. For (b), see [29, Lemma 8.4], which follows the geometric argument of Tao and Vu [81] (in [29] the lemma is stated for the case that G is a real Ginibre matrix, but the same argument gives the above statement). □

We also note that the unshifted matrix $B(t)$ is always well invertible with high probability:

Lemma 4.4. *For any $0 < t \leq T < \infty$ and $\delta > 0$,*

$$\mathbb{P}(\sigma_N(B(t)) \leq \delta) = O_{\varrho,T}(\delta)^{N^{2/3}}.$$

Proof. For any $t > 0$ we have

$$\mathbb{P}\left(\sigma_N\left(B(t)\right) \leq \delta\right) = \mathbb{P}\big(\big\|B(t)^{-1}\big\| \geq 1/\delta\big).$$

However, following (1.5), if we define $(\mathcal{B}(t))_{t\geq 0}$ as the solution of the following stochastic differential equation:

$$\mathcal{B}(t) = I - \int_0^t dW(s)\mathcal{B}(s) + \frac{\zeta}{2}\int_0^t \mathcal{B}(s)ds$$

we get thanks to classical stochastic calculus that $B(t)\mathcal{B}(t) = I$ for all $t \geq 0$ (see [59, Section 4.2] for further details). In particular,

$$\mathbb{P}(\sigma_N\left(B(t)\right) \leq \delta) = \mathbb{P}\left(\|\mathcal{B}(t)\| \geq 1/\delta\right) = \mathbb{P}\left(\|\mathcal{B}(t)^*\| \geq 1/\delta\right).$$

But if $\widetilde{B}(t) = ((B(t))^{-1})^*$, and $\widetilde{W}(t) = -\left(W(t)\right)^*$, we have that

$$\widetilde{B}(t) = I + \int_0^t \widetilde{B}(s)d\widetilde{W}(s) + \frac{\overline{\zeta}}{2}\int_0^t \widetilde{B}(s)ds.$$

Since $(\widetilde{W}(t))_{t\geq 0}$ has the same law as $(W_{\varrho,\overline{\zeta}}(t))_{t\geq 0}$, we have that $(\widetilde{B}(t))_{t\geq 0}$ is a $(\varrho,\overline{\zeta})$-Brownian motion. Hence the conclusion follows thanks to Theorem 3.6. □

We turn to prove Proposition 4.1. Fix $0 < t \leq T < \infty$ and $\kappa > 0$. By our assumption that $|\zeta| < \varrho$ we may assume without loss of generality that $0 < b \leq |a|$.

Given $(B(t))_{t\geq 0}$ and $(\widetilde{B}(t))_{t\geq 0}$ independent (ϱ,ζ)-Brownian motions driven respectively by $(W(t))_{t\geq 0}$ and $(\widetilde{W}(t))_{t\geq 0}$, we have for any $0 \leq \varepsilon \leq t$ the equality in law

$$B(t) \stackrel{d}{=} B(t-\varepsilon)\widetilde{B}(\varepsilon). \tag{4.3}$$

Next we show that up to a small error we can replace $\widetilde{B}(\varepsilon)$ with a shift $cG + M$ of a scaled Ginibre matrix G for some $c = c(\varepsilon) > 0$, where G and M are independent. Indeed, note that if $\alpha, \beta \in \mathbb{R}$ and X, Y are independent GUE random matrices of size N, then $\alpha X + \beta Y$ has the same law as $\sqrt{\alpha^2+\beta^2}Z$ where Z is also a GUE random matrix. Furthermore, $\widetilde{W}(\varepsilon)$ has the same law as

$$\sqrt{\varepsilon}e^{\mathbf{i}\theta}\left(aX + \mathbf{i}bY\right).$$

Thus, since $|a| \geq b > 0$, we get that $\widetilde{W}(\varepsilon)$ has the same law as

$$b\sqrt{\varepsilon}e^{\mathbf{i}\theta}\left(X + \mathbf{i}Y\right) + \sqrt{|\zeta|}\sqrt{\varepsilon}e^{\mathbf{i}\theta}\widetilde{X}, \tag{4.4}$$

where $\widetilde{X}$ is an independent copy of X. Consequently,

$$\widetilde{W}(\varepsilon) \stackrel{d}{=} b\sqrt{2\varepsilon}G + \sqrt{|\zeta|}\sqrt{\varepsilon}e^{\mathbf{i}\theta}\widetilde{X} \tag{4.5}$$

where G is a Ginibre random matrix independent of $\widetilde{X}$.

For $\varepsilon, \delta_0, \delta_1 \geq 0$, denote the event

$$\mathcal{G} = \mathcal{G}(\delta_0, \delta_1, \varepsilon) := \left\{ \sigma_N(B(t-\varepsilon)) \geq \delta_0,\ \|\widetilde{B}(\varepsilon) - I - \widetilde{W}(\varepsilon)\| \leq \delta_1 \right\}. \tag{4.6}$$

From Lemma 4.4 and Theorem 1.7,

$$\mathbb{P}(\mathcal{G}(\delta_0, \delta_1, \varepsilon)^c) \leq O_{\varrho,T}(\delta_0)^{N^{2/3}} + O_{\varrho,T}(\varepsilon/\delta_1)^{N^{2/3}}. \tag{4.7}$$

On $\mathcal{G} \cap \{\sigma_N(B_0) \geq \kappa\}$, for any $0 \leq \ell \leq N-1$ we have

$$\sigma_{N-\ell}(B_0 B(t-\varepsilon)\widetilde{B}(\varepsilon) + Z) \geq \kappa\delta_0 \sigma_{N-\ell}\big(\widetilde{B}(\varepsilon) + B(t-\varepsilon)^{-1}B_0^{-1}Z\big)$$

and

$$\sigma_{N-\ell}\big(\widetilde{B}(\varepsilon) + B(t-\varepsilon)^{-1}B_0^{-1}Z\big) \geq \sigma_{N-\ell}\big(\widetilde{W}(\varepsilon) + I + B(t-\varepsilon)^{-1}B_0^{-1}Z\big) - \delta_1.$$

Combining with (4.3) and (4.5) we have that for any $\delta \geq 0$,

$$\begin{aligned}
&\mathbb{P}\Big(\sigma_{N-\ell}(B_0 B(t) + Z) \leq \delta\,,\ \sigma_N(B_0) \geq \kappa\,,\ \mathcal{G}\Big) \\
&\quad = \mathbb{P}\Big(\sigma_{N-\ell}(B_0 B(t-\varepsilon)\widetilde{B}(\varepsilon) + Z) \leq \delta\,,\ \sigma_N(B_0) \geq \kappa\,,\ \mathcal{G}\Big) \\
&\quad \leq \mathbb{P}\Big(\sigma_{N-\ell}\big(b\sqrt{2\varepsilon}G + M\big) \leq \frac{\delta}{\kappa\delta_0} + \delta_1\Big)
\end{aligned} \tag{4.8}$$

where

$$M := I + \sqrt{|\zeta|}\sqrt{\varepsilon}e^{\mathbf{i}\theta}\widetilde{X} + B(t-\varepsilon)^{-1}B_0^{-1}Z$$

is independent of the Ginibre matrix G.

To conclude the proof of Proposition 4.1(a) we use Lemma 4.3(a), along with (4.7) and (4.8) with $\ell = 0$ and $\varepsilon, \delta_0, \delta_1 > 0$ to be chosen, to get

$$\mathbb{P}\Big(\sigma_N(B_0 B(t) + Z) \leq \delta\,,\ \sigma_N(B_0) \geq \kappa\Big) \leq \frac{N^2}{2b^2\varepsilon}\left(\frac{\delta}{\kappa\delta_0} + \delta_1\right)^2 + O_{\varrho,T}(\delta_0 + \varepsilon\delta_1^{-1})^{N^{2/3}}.$$

The claim follows upon taking $\varepsilon := \delta$, $\delta_0 := \delta^{\alpha/2}$ and $\delta_1 := \delta^{1-\alpha/2}$.

Turning to conclude the proof of Proposition 4.1(b), from a union bound over the choices of ℓ, it suffices to show that for any fixed $\ell \geq C\log N$,

$$\mathbb{P}(\sigma_{N-\ell}(B_0 B(t) + Z) \leq c(\ell/N)^2) \leq O(Ne^{-\ell}) + \exp(-2N^{2/3}). \tag{4.9}$$

Taking $\delta_0 = c$, $\delta = \delta_1 = c^2(\ell/N)^2$, and $\varepsilon = c\delta$ for sufficiently small $c = c(T, \varrho, \zeta, \kappa) > 0$, from (4.8) and (4.7) we can bound the left hand side above by

$$\mathbb{P}\left(\sigma_{N-\ell}\Big(G + \frac{1}{b\sqrt{2\varepsilon}}M\Big) \leq \frac{2\sqrt{c}}{\kappa b}\frac{\ell}{N}\right) + \exp(-2N^{2/3}).$$

Taking c smaller if necessary, Proposition 4.1(b) now follows from Lemma 4.3(b).

5. An Analytic Approach for Mesoscopic Singular Values

In this section we provide an alternative argument for a lower bound on small singular values of $B_0^N B^N(t) - zI$, as in Proposition 4.1(b). That proposition implies that the empirical cumulative distribution function of shifted singular values is essentially Hölder-$\frac{1}{2}$ continuous near 0, at shrinking scales, for all $z \in \mathbb{C}$. This is enough (in conjunction with the bound on polynomial decay of the smallest singular value in 4.1(a)) to yield convergence of the empirical law of eigenvalues; in fact, it would be enough to have a Hölder continuity bound (of varying exponent) just at Lebesgue almost every $z \in \mathbb{C}$.

In this section, we show that: at least for the Brownian motion $B^N(t)$ started at the identity (i.e. $B_0^N = I$), the expected empirical CDF of singular values $\sigma_j(B^N(t) - zI)$ is actually Lipschitz near 0, for Lebesgue almost all $z \in \mathbb{C}$. In particular: the only z where this may fail (and instead we get Hölder-$\frac{1}{2}$ continuity) are on the boundary of the the support set of the Brown measure of the limit free multiplicative Brownian motion.

To state and prove this precisely, we make the same parameter reduction as in Notation 1.12: Since $B^N_{\varrho,\zeta}(t) \stackrel{\mathrm{d}}{=} B^N_{t\varrho,t\zeta}(1)$, we henceforth consider the random matrices

$$B^N(t,\zeta) = B^N_{t,\zeta}(1) \quad \& \quad b(t,\zeta) = b_{t,\zeta}(1). \tag{5.1}$$

Recall that the Brown measure of $b(t,\zeta)$ was identified in [50]; in particular, its support set is a compact region $\overline{\Sigma}(1,t,\zeta)$ defined in Theorem 1.13. The boundary $\partial\Sigma(1,t,\zeta)$ is a simple smooth closed curve if $t < 4$, a smooth "figure-eight" curve if $t = 4$, and the union of two disjoint simple smooth closed curves if $t > 4$. In particular, the Lebesgue measure of $\partial\Sigma(1,t,\zeta)$ is 0 for all $t > 0$ and $|\zeta| < t$.

Proposition 5.1. *Let $T > 0$ and $\zeta \in \mathbb{C}$ with $|\zeta| < t$. Let $z \in \mathbb{C} \setminus \partial\Sigma(1,t,\zeta)$. Then there is a constant $\overline{C}_{z,T}$ so that, for all $N \in \mathbb{N}$, all $t \in [0,T]$, and all $\eta > N^{-2/11}$,*

$$\left|\operatorname{Im} \mathbb{E} \operatorname{tr}_N[(i\eta - |B^N(t,\zeta) - zI|)^{-1}]\right| \leq \overline{C}_{z,T}. \tag{5.2}$$

Proposition 5.1 is a strong *Wegner anti-concentration estimate* (cf. [87]). The (conclusion of the) proof is in Section 5.3 below. It follows that the expected CDF of the empirical law of singular values of $B^N_{\varrho,\zeta}(t) - zI$ is uniformly Lipschitz on mesoscopic-to-macroscopic scales.

Corollary 5.2. *For any $0 < t \leq T < \infty$ and $z \in \mathbb{C} \setminus \partial\Sigma(1, t\varrho, t\zeta)$, we have*

$$\mathbb{E}\mu_{|B^N_{\varrho,\zeta}(t) - zI|}([0,\eta]) = O_{\varrho,z,T}(\eta)$$

uniformly for $\eta \in [N^{-2/11}, 1]$.

Proof. Since $B^N_{\varrho,\zeta}(t) \stackrel{\mathrm{d}}{=} B^N_{t\varrho,t\zeta}(1) = B^N(t\varrho, t\zeta)$, it suffices to prove the corollary for $B^N(t,\zeta)$; i.e. for $z \in \mathbb{C} \setminus \partial\Sigma(1,t,\zeta)$,

$$\mathbb{E}\mu_{|B^N(t,\zeta) - zI|}([0,\eta]) = O_{z,T}(\eta). \tag{5.3}$$

Now since $\operatorname{Im}(i\eta - x)^{-1} = -\eta/(\eta^2 + x^2)$ for $x \in \mathbb{R}$, the claim follows from Proposition 5.1 and the pointwise bound $\eta/(\eta^2 + x^2) \geq (2\eta)^{-1} 1_{x \in [0,\eta]}$. □

Corollary 5.2 can be substituted for Proposition 4.1(b) in the proof of Theorem 1.6 in Section 6 (for the case $B_0^N = I$), following the argument from [48].

Remark 5.3. *It is worth noting that, for any matrix $A^N \in \mathbb{M}_N(\mathbb{C})$, the spectral measure $\mu_{|A^N|}$ is exactly the empirical law of singular values $\sigma_j(A^N)$; thus*

$$\mu_{|A^N|}([0,\eta]) = \frac{1}{N}\sum_{j=1}^{N} \delta_{\sigma_j(A^N)}([0,\eta]) = \frac{\#\{j \leq N : \sigma_j(A^N) \leq \eta\}}{N}.$$

This is indeed the empirical CDF of singular values. Hence, Corollary 5.2 says that the expected empirical CDF of $B^N_{\varrho,\zeta}(t) - zI$ is Lipschitz near 0, with a Lipschitz constant that is is locally uniform in t, for all z except possibly on the boundary of the domain $\Sigma(1, t\varrho, t\zeta)$.

The quantity on the left-hand-side of (5.2) is the (imaginary part of the) expected Cauchy transform $G_{|B^N(t,\zeta)-zI|}(i\eta)$. Our approach to prove Proposition 5.1 is to compare this to the Cauchy transform of the large-N limit operator, $G_{|b(t,\zeta)-z|}(i\eta)$. As shown in Section 5.1 below, this difference is $O(1/N^2)$ with constants that depend polynomially on $1/\eta$; then by choosing $\eta > N^{-\alpha}$ for an appropriate power $\alpha > 0$, the bound (5.2) then reduces to a similar bound on the free multiplicative Brownian motion; i.e. it reduces to showing $G_{|b(t,\zeta)-z|}(i\eta)$ is bounded for $\eta > 0$. This is accomplished in Section 5.2, primarily using analytic results derived from the PDE methods in [37, 50] together with spectral bounds from [51].

5.1. **Smooth Functional Comparison to the Large-N Limit.**

Throughout this section, our estimates will be formulated in terms of the over-parametrized $B^N(t) = B^N_{\varrho,\zeta}(t)$; they will be converted to the reduced parametrization $B^N(t,\zeta)$ when needed in Section 5.3. Additionally, the tools in this section apply equally well for a deterministic initial condition B_0^N, so we prove them in that level of generality for potential future use.

We begin with the following lemma.

Lemma 5.4. *Let B_0^N be a deterministic matrix. There exists a constant $C_{\varrho,t}$ such that for any $y \in \mathbb{R}$, we have that*

$$\begin{aligned}&\left|\mathbb{E}\left[\mathrm{tr}_N\left(e^{\mathbf{i}y|B_0^N B^N(t)-z|^2}\right)\right] - \tau_N\left(e^{\mathbf{i}y|B_0^N b(t)-z|^2}\right)\right| \\ &\leq C_{\varrho,t}\frac{(1+y^4)(1+|z|^4)(1+\left\|B_0^N\right\|^4)}{N^2},\end{aligned}$$

where we used the construction of the space $\mathscr{A}_N$ of Definition 2.1 to define the product of B_0^N and $b(t)$.

Proof. Given two operators A and B, one has that

$$e^A - e^B = \int_0^1 e^{\alpha A}(A-B)e^{(1-\alpha)B}d\alpha.$$

Consequently,

$$\begin{aligned}&\left|\mathbb{E}\left[\mathrm{tr}_N\left(e^{\mathbf{i}y|B_0^N B^N(t)-z|^2}\right)\right] - \mathbb{E}\left[\mathrm{tr}_N\left(e^{\mathbf{i}y|B_0^N B_n^N(t)-z|^2}\right)\right]\right| \\ &= \left|y\int_0^1 \mathbb{E}\Big[\mathrm{tr}_N\Big(e^{\mathbf{i}\alpha y|B_0^N B^N(t)-z|^2}\Big(|B_0^N B^N(t)-z|^2 \right. \\ &\qquad\qquad \left. - |B_0^N B_n^N(t)-z|^2\Big)e^{\mathbf{i}y(1-\alpha)|B_0^N B_n^N(t)-z|^2}\Big)\Big]d\alpha\right| \\ &\leq |y|\left\|B_0^N\right\|\left(\left\|B_0^N B^N(t)-z\right\|_2 + \left\|B_0^N B_n^N(t)-z\right\|_2\right)\left\|B^N(t)-B_n^N(t)\right\|_2,\end{aligned}$$

where we used that for any self-adjoint operator X, the operator norm of $e^{\mathbf{i}yX}$ is equal to 1. Thus thanks to Lemma 3.1 and Proposition 3.3,

$$\mathbb{E}\left[\mathrm{tr}_N\left(e^{\mathbf{i}yB_0^N B^N(t)}\right)\right] = \lim_{n\to\infty}\mathbb{E}\left[\mathrm{tr}_N\left(e^{\mathbf{i}yB_0^N B_n^N(t)}\right)\right].$$

And similarly,

$$\tau_N(e^{\mathbf{i}yB_0^N b(t)}) = \lim_{n\to\infty}\tau_N(e^{\mathbf{i}yB_0^N b_n(t)}).$$

Finally, thanks to Lemma 3.6 of [74], one has that

$$\begin{aligned}&\mathbb{E}\left[\mathrm{tr}_N\left(e^{\mathbf{i}y\left|B_0^N B^N(t)-z\right|^2}\right)\right]-\tau_N\left(e^{\mathbf{i}y|B_0^N b(t)-z|^2}\right)\\&=\frac{1}{N^2}\int_0^\infty\int_0^u e^{-v-u}\int_{[0,1]^4}\mathbb{E}\left[\tau_N\left(R^{n,N,y}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,}\right)\right]\,d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 dv du,\end{aligned}$$

where

$$\frac{\left|\mathbb{E}\left[\tau_N\left(R^{n,N,y}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,}\right)\right]\right|}{(1+y^4)(1+|z|^4)(1+\left\|B_0^N\right\|^4)}$$

can, up to a universal constant, be upper bounded by

$$\sup_{1\le a\le b\le\lfloor 2^n t\rfloor}\mathbb{E}\left[\mathrm{tr}_N\left(\left|\prod_{i=a}^b\left(I_N+\frac{e^{\mathbf{i}\theta}}{2^{n/2}}\left(aX^{N,\{v,u\}}_{i,\{1,2\}}+\mathbf{i}bY^{N,\{v,u\}}_{i,\{1,2\}}\right)+\frac{\zeta}{2^{n+1}}I_N\right)\right|^{12}\right)\right],$$

where the variables $X^{N,\{v,u\}}_{i,\{1,2\}}$ and $Y^{N,\{v,u\}}_{i,\{1,2\}}$ are defined as in Lemma 2.25. Besides, by the same proof as in Lemma 3.1, the quantity above is bounded by a constant which only depends on t and ϱ since $|\zeta|\le\varrho$. Hence the conclusion. □

The following corollary is the main tool to handle the global regime.

Corollary 5.5. *There exists a constant $C_{\varrho,t}$ such that for any functions $f\in\mathcal{C}^5(\mathbb{R})$ with compact support bounded by K, we have that*

$$\begin{aligned}&\left|\mathbb{E}\left[\mathrm{tr}_N\left(f\left(\left|B_0^N B^N(t)-z\right|^2\right)\right)\right]-\tau_N\left(f\left(|B_0^N b(t)-z|^2\right)\right)\right|\\&\le C_{\varrho,t}\sqrt{K}\frac{(1+|z|^4)(1+\left\|B_0^N\right\|^4)\left\|f\right\|_{\mathcal{C}^5(\mathbb{R})}}{N^2},\end{aligned}\tag{5.4}$$

where $\|f\|_{\mathcal{C}^5(\mathbb{R})}$ is the sum of the supremum over $\mathbb{R}$ of the k-th derivative of f with k from 0 to 5.

Proof. Let $f\in L^1(\mathbb{R})$, we set

$$\widehat{f}(y):=\frac{1}{2\pi}\int_{\mathbb{R}}e^{-\mathbf{i}yx}f(x)dx.$$

Then if $\widehat{f}\in L^1(\mathbb{R})$, by Fourier inversion theorem,

$$f(x)=\int_{\mathbb{R}}e^{\mathbf{i}xy}\widehat{f}(y)dy.$$

Besides, if f is sufficiently smooth, then $\left|y^k\widehat{f}(y)\right|=\left|\widehat{f^{(k)}}(y)\right|$, where $f^{(k)}$ is the k-th derivative of f. Consequently,

$$\begin{aligned}&\int_{\mathbb{R}}|y^k|\left|\widehat{f}(y)\right|dy\\&=\int_{\mathbb{R}}\frac{(|y^k|+|y^{k+1}|)\left|\widehat{f}(y)\right|}{1+|y|}dy\\&\le\int_{\mathbb{R}}\frac{\left|\widehat{f^{(k)}}(y)\right|+\left|\widehat{f^{(k+1)}}(y)\right|}{1+|y|}dy\\&\le\left(\left(\int_{\mathbb{R}}\left|\widehat{f^{(k)}}(y)\right|^2dy\right)^{1/2}+\left(\int_{\mathbb{R}}\left|\widehat{f^{(k+1)}}(y)\right|^2dy\right)^{1/2}\right)\left(\int_{\mathbb{R}}\frac{1}{(1+|y|)^2}dy\right)^{1/2}\end{aligned}$$

$$\leq \frac{1}{\sqrt{2\pi}} \left(\left(\int_{\mathbb{R}} \left| f^{(k)}(y) \right|^2 dy \right)^{1/2} + \left(\int_{\mathbb{R}} \left| f^{(k+1)}(y) \right|^2 dy \right)^{1/2} \right) \left(\int_{\mathbb{R}} \frac{1}{(1+|y|)^2} dy \right)^{1/2}$$

$$\leq \sqrt{K} \left(\frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{(1+|y|)^2} dy \right)^{1/2} \left(\sup_{y \in \mathbb{R}} \left| f^{(k)}(y) \right| + \sup_{y \in \mathbb{R}} \left| f^{(k+1)}(y) \right| \right),$$

where we used Plancherel theorem to go from the third to the fourth line. Thus, we have that

$$\int_{\mathbb{R}} (1+y^4) \left| \widehat{f}(y) \right| dy \leq \sqrt{K} \left(\frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{(1+|y|)^2} dy \right)^{1/2} \|f\|_{\mathcal{C}^5(\mathbb{R})}.$$

Hence the conclusion by using Lemma 5.4. □

5.2. **Free Wegner Estimate.**

Here we prove a version of the desired Wegner estimate in Proposition 5.1 but for the free multiplicative Brownian motion already in the large-N limit. The reader may wish to review Section 1.6 for the definitions of the regions $\Sigma(1,t) = \Sigma(1,t,0)$ and more generally $\Sigma(1,t,\zeta)$; here the 1 is the initial condition $b_0 = u_0 = 1$ for the Brownian motion.

Lemma 5.6. *Let $T > 0$ and $\zeta \in \mathbb{C}$ with $|\zeta| < t$. Let $z \in \mathbb{C} \setminus \partial\Sigma(1,t,\zeta)$. Then there is a constant $\overline{C}_{z,T}$ so that, for all $t \in [0,T]$, and all $\eta > 0$,*

$$\left| \operatorname{Im} \tau[(i\eta - |b(t,\zeta) - z|)^{-1}] \right| \leq \overline{C}_{z,T}. \tag{5.5}$$

Proof. The potentially bad set $\partial\Sigma(1,t,\zeta)$, which is of Lebesgue measure 0, separates $\mathbb{C}$ into two regions. We treat separately the two cases: where $z \in \Sigma(1,t,\zeta)$, and where $z \notin \overline{\Sigma(1,t,\zeta)}$. We begin with the latter.

In [51, Theorem 3.7], it was shown that $\overline{\Sigma(1,t,\zeta)}$ is precisely equal to the spectrum of $b(t,\zeta)$. Hence, for $z \notin \overline{\Sigma(1,t,\zeta)}$, $b(t,\zeta) - z$ has a bounded inverse, and therefore so does $|b(t,\zeta) - z|$ (by the spectral theorem). For readability, set $x = |b(t,\zeta) - z|$. We can therefore compute that

$$(i\eta - x)^{-1} = -x^{-1}(1 - i\eta x^{-1})^{-1} = -x^{-1} \sum_{k=0}^{\infty} (ix^{-1})^k \eta^k$$

which converges in $\mathscr{A}$ provided $|\eta| < \|x^{-1}\|^{-1}$. Thus

$$\tau[(i\eta - |b(t,\zeta) - z|)^{-1}] = -\sum_{k=0}^{\infty} i^k \tau[(x^{-1})^{k+1}] \eta^k$$

has a convergent power series expansion for $|\eta| < \||b(t,\zeta) - z|^{-1}\|^{-1}$, and hence it is bounded for $0 < \eta < \||b(t,\zeta) - \lambda|^{-1}\|^{-1}$. Since this norm is continuous in t, the bound is locally uniform in T, as states.

Now, consider the case that $z \in \Sigma(1,t,\zeta)$. Define

$$S(t,z,\zeta,\eta) = \tau[|\log(|b(t,\zeta) - z|^2 + \eta^2)].$$

(Then $\lim_{\eta \downarrow 0} S(t,z,\zeta,\eta) = 4\pi L_{b(t,\zeta)}(z)$, and the Brown measure of $b(t,\zeta)$ is the Laplacian of this limit.) Notice that

$$\frac{\partial}{\partial \eta} S(t,z,\zeta,\eta) \Big|_{\eta=\eta_0} = 2\eta_0 \, \tau[(|b(t,\zeta) - z|^2 + \eta_0^2)^{-1}] = -2\operatorname{Im} \tau[(i\eta_0 - |b(t,\zeta) - z|)^{-1}]$$

where the second equality follows just as in Corollary 5.2 from the fact that $\mathrm{Im}(i\eta - x)^{-1} = -\eta/(\eta^2 + x^2)$ for $x \in \mathbb{R}$.

Now, [50, Theorem 8.2] (or more precisely its proof) showed that there is a diffeomorphism $\Phi_\zeta \colon \Sigma(1,t) \to \Sigma(1,t,\zeta)$ such that

$$S(t,z,\zeta,\eta) = S(t, \Phi_\zeta^{-1}(z), 0, \eta).$$

Moreover, the diffeomorphism does not depend on t or η. Hence

$$\mathrm{Im}\,\tau[(i\eta_0 - |b(t,\zeta) - z|)^{-1}] = -\frac{1}{2}\frac{\partial}{\partial\eta} S(t,z,\zeta,\eta)\Big|_{\eta=\eta_0} = \frac{\partial}{\partial\eta} S(t, \Phi_\zeta^{-1}(z), \eta)\Big|_{\eta=\eta_0}$$

By [37, Theorem 6.4], for $z' \in \Sigma(1,t)$ the function $(t,\eta) \mapsto S(t,z',0,\eta)$ has real analytic continuation to $\mathbb{R}_+ \times (-\delta, \delta)$ where δ may depend on z, but not t. (The statement of that theorem suggests the analytic continuation is only local in t as well; however, in the proof, the actual extension implicitly computed in [37, Eq. (6.20)], is analytic on $\mathbb{R}_+$ and continuous on $[0,\infty)$.) Since $z' = \Phi_\zeta^{-1}(z) \in \Sigma(1,t)$ precisely when $z \in \Sigma(1,t,\zeta)$, it therefore follows that the $\frac{\partial}{\partial\eta}$ derivative above is continuous and bounded, locally uniformly in t, for $0 < \eta_0 < \delta/2$ (where δ may depend on z).

Hence, for such z, the derivative $\frac{\partial}{\partial\eta} S(t,z,\zeta,\eta)$ is continuous and bounded for $0 < \eta < \delta/2$. What's more, the Cauchy transform $|[(i\eta - |b(t,\zeta) - z|)^{-1}]|$ is $\leq 2/\delta$ for all $\eta \geq \delta/2$. This proves the bound for all $\eta > 0$, concluding the proof. □

5.3. **Proof of Proposition 5.1.**

We now combine Corollary 5.5 with Lemma 5.5 to prove Proposition 5.3.

Taking $B_0^N = I$ and reparametrizing per (5.1), (5.4) yields

$$\begin{aligned} &\left|\mathbb{E}\left[\mathrm{tr}_N\left(f(|B^N(t,\zeta) - zI|^2)\right)\right] - \tau\left(f(|b(t,\zeta) - z|^2)\right)\right| \\ &\leq 2C_{t\varrho,t}\sqrt{K}\frac{(1+|z|^4)\|f\|_{\mathcal{C}^5(\mathbb{R})}}{N^2} \end{aligned} \tag{5.6}$$

for any function $f \in \mathcal{C}^5(\mathbb{R})$ that is compactly supported with support of diameter $\leq K$. Let $M = M(t,z,\zeta) < \infty$ be a constant large enough that $\||B^N(t,\zeta) - zI|^2\| \leq M$ a.s. for all large N (cf. Corollary 3.7) and $\||b(t,\zeta) - z|^2\| \leq M$. For given $\eta \in (0,1]$, let f_η be a smooth, compactly supported function satisfying

- $f_\eta(x) = -\frac{\eta}{\eta^2 + x}$ for $x \in [0, M]$
- For each $k \leq 5$, the maximum derivative $\max_{x\in\mathbb{R}} |f_\eta^{(k)}(x)|$ does not exceed the maximum derivative $\max_{x\in[0,M]} |f_\eta^{(k)}(x)| = |f_\eta^{(k)}(0)| = \frac{k!\eta}{(\eta^2)^{k+1}} = k!\eta^{-2k-1}$.

This can be accomplished with a smooth function f_η whose support diameter K is uniform in $\eta \in (0,1]$: the smaller $\eta > 0$ is, the larger the maximum derivative of the approximation is allowed to be. Hence, $K = K(M)$ can be taken to depend on M independent of η.

Then, since the spectra of $|B^N(t,\zeta) - zI|^2$ and $|b(t,\zeta) - z|^2$ are contained in the interval $[0, M]$, it follows that

$$\begin{aligned} \mathrm{tr}_N\left(f(|B^N(t,\zeta) - zI|^2)\right) &= -\lambda\,\mathrm{tr}_N\left((\eta^2 + |B^N(t,\zeta) - zI|^2)^{-1}\right) \\ &= \mathrm{Im}\ \mathrm{tr}_N[(i\eta - |B^N(t,\zeta) - zI|)^{-1}]. \\ \&\qquad \tau\left(f(|b(t,\zeta) - z|^2)\right) &= -\lambda\tau\left((\eta^2 + |b(t,\zeta) - z|^2)^{-1}\right) \end{aligned}$$

$$= \operatorname{Im} \tau[(i\eta - |b(t,\zeta) - z|)^{-1}].$$

What's more, the second condition on f_η shows that

$$\|f_\eta\|_{\mathcal{C}^5} = \sum_{k=0}^{5} k! \eta^{-2k-1} \leq 154\eta^{-11}.$$

Hence, (5.6) shows that

$$\begin{aligned} &\left|\mathbb{E}\left(\operatorname{Im} \operatorname{tr}_N[(i\eta - |B^N(t,\zeta) - zI|)^{-1}]\right) - \operatorname{Im} \tau[(i\eta - |b(t,\zeta) - z|)^{-1}]\right| \\ &\leq 2C_{t\varrho,t}\sqrt{K(M)}\frac{(1+|z|^4)\cdot 154\eta^{-11}}{N^2}. \end{aligned}$$

That is to say: for all $N \in \mathbb{N}$ and all $\eta \in (0,1]$,

$$\begin{aligned} &\left|\mathbb{E}\left(\operatorname{Im} \operatorname{tr}_N[(i\eta - |B^N(t,\zeta) - zI|)^{-1}]\right)\right| \\ &\leq \left|\operatorname{Im} \tau[(i\eta - |b(t,\zeta) - z|)^{-1}]\right| + 308C_{t\varrho,t}\sqrt{K(M)}(1+|z|^4)\frac{\eta^{-11}}{N^2}. \end{aligned} \tag{5.7}$$

By Lemma 5.6, the first term is bounded above by a constant $\overline{C}_{z,T}$ for all $\eta > 0$ and for all $t \in [0,T]$. For the second term, if we take $\eta \geq N^{-2/11}$ then $\eta^{-11}/N^2 \leq 1$. This proves boundedness.

Lastly, we consider the dependence of the bound on t. The constant $K(M)$ depends on z, ζ, t only through the uniform upper bound on $\||B^N(t,\zeta) - zI|^2\| \leq \|B^N(t,\zeta)\|^2 + 2|z|\|B^N(t,\zeta)\| + |z|^2$. The strong convergence result of [11] shows that $\|B^N(t,\zeta)\|$ is bounded uniformly in N by a smooth exponential function of t. Finally, the constant $C_{t\varrho,t}$ is increasing in t, as is clear from the last display equation in the proof of Lemma 5.4 defining it. This shows, therefore, that the bound in Proposition 5.1 is indeed locally uniform in t, concluding the proof.

6. Proof of Theorem 1.6: Convergence of Eigenvalues

6.1. **Almost sure convergence of the eigenvalues.** We recall the following lemma from [20] encapsulating the Hermitization procedure outlined in Section 1.5.

Lemma 6.1 ([20, Lemma 4.3])**.** *Let $(A_N)_{N\geq 1}$ be a sequence of random matrices where A_N is $N \times N$. Suppose there exists a family of non-random probability measures $(\nu_z)_{z\in\mathbb{C}}$ on $\mathbb{R}_+$ such that for a.e. $z \in \mathbb{C}$, almost surely:*

(a) $\mu_{|A^N - zI_N|} \to \nu_z$, and
(b) $\log$ is uniformly integrable for $(\mu_{|A^N - zI_N|})_{N\geq 1}$.

Then there exists a probability measure μ on $\mathbb{C}$ such that

(a') $\mu_{A^N} \to \mu$ weakly almost surely, and
(b') for a.e. $z \in \mathbb{C}$, we have

$$\int_{\mathbb{C}} \log|\lambda - z| d\mu(\lambda) = -\int_0^\infty \log(s) d\nu_z(s).$$

Lemma 1.9 verifies condition (a) for $A^N = B_0^N B^N(t)$, with $(\nu_z)_{z\in\mathbb{C}}$ as in (1.19) satisfying (b') with $\mu = \mu_{b_0,t}$.

It only remains to verify condition (b). From the Borel–Cantelli lemma it suffices to show that for every $\varepsilon > 0$ and a.e. $z \in \mathbb{C}$, there exists $L = L(\varepsilon, z) < \infty$ such that

$$\mathbb{P}\bigg(\int_{\{|\log s| > L\}} |\log s| d\mu_{|B_0^N B^N(t) - z|}(s) > \varepsilon \bigg) \leq N^{-2} \tag{6.1}$$

for all N sufficiently large. We show this holds for all $z \in \mathbb{C}$.

Fix $z \in \mathbb{C}$ arbitrary. We henceforth suppress from the notation the dependence of constants on the fixed parameters $\kappa, \varrho, \zeta, T, z$. From Theorem 3.6 and our assumption on B_0^N, we have $\|B_0^N B^N(t) - z\| = O(1)$ with probability at least $1 - \exp(-N^{2/3})$. It thus suffices to show

$$\int_0^\delta |\log s| d\mu_{|B_0^N B^N(t) - z|}(s) \lesssim o(1) + f(\delta) \tag{6.2}$$

for some function $f(\delta) = o_{\delta \to 0}(1)$.

For brevity we denote the singular values of $B_0^N B^N(t) - z$ by $\sigma_i := \sigma_i(B_0^N B^N(t) - z)$. From Proposition 4.1, there is an event $\mathcal{G}$ of probability at least $1 - N^{-5}$ (say) such that on $\mathcal{G}$,

$$\sigma_N \geq N^{-10} \tag{6.3}$$

and

$$\sigma_{N-k} \geq c(k/N)^2 \qquad \forall\, k \geq C \log N \tag{6.4}$$

for some constants $c, C > 0$. For any $\delta \in (0,1)$, on $\mathcal{G}$ we have

$$\begin{aligned}
&\int_0^\delta \log(1/s) d\mu_{|B_0^N B^N(t) - z|}(s) \\
&= \frac{1}{N} \sum_{k=0}^{N-1} \log \frac{1}{\sigma_{N-k}} 1_{\sigma_{N-k} \leq \delta} \\
&\leq C \frac{\log N}{N} \log \frac{1}{\sigma_N} + \frac{1}{N} \sum_{k \geq C \log N} \log \frac{1}{\sigma_{N-k}} 1_{\sigma_{N-k} \leq \delta} \\
&\lesssim \frac{\log^2 N}{N} + \frac{1}{N} \sum_{k \leq (\delta/c)^{1/2} N} \log \frac{N}{k} \\
&\lesssim \frac{\log^2 N}{N} + \int_0^{\sqrt{\delta/c}} \log(1/s) ds
\end{aligned}$$

where in the final line we recognized the sum over k as a Riemann sum for the integral of log near the origin. Since log is locally integrable under the Lebsegue measure, we hence obtain (6.2) to conclude the proof of Theorem 1.6.

6.2. **An estimate on the expectation of the Brown measure.** Let us Begin with the following Lemma that allows us to control the global regim.

Lemma 6.2. *Given f_ε a smooth function taking value in $[0,1]$ such that $f_{|(-\infty, \varepsilon/2]} = 0$ and $f_{|[\varepsilon, \infty)} = 0$, we set*

$$g_\varepsilon(t) := \begin{cases} \ln(t) f_\varepsilon(t) & \text{if } t > 0, \\ 0 & \text{else.} \end{cases}$$

Then there exists a constant $C_{\varrho,t}$ such that for $\varepsilon < 1/2$,

$$\left|\mathbb{E}\left[\mathrm{tr}_N\left(g_\varepsilon\left(\left|B_0^N B^N(t) - z\right|^2\right)\right)\right] - \tau_N\left(g_\varepsilon\left(|B_0^N b(t) - z|^2\right)\right)\right| \leq C_{\varrho,t}\frac{(1+|z|)^5(1+\left\|B_0^N\right\|)^5|\ln(\varepsilon)|}{\varepsilon^5 N^2}.$$

Proof. We set $\widetilde{M} = \left\|B_0^N\right\| M + |z|$ and h a smooth function taking value in $[0,1]$ and such that $h_{|(-\infty,\widetilde{M}]} = 0$, $h_{|[\widetilde{M}+1,\infty)} = 1$, then

$$\mathbb{E}\left[\mathrm{tr}_N\left(g_\varepsilon\left(\left|B_0^N B^N(t) - z\right|^2\right)\right)\right] = \mathbb{E}\left[\mathrm{tr}_N\left((g_\varepsilon h)\left(\left|B_0^N B^N(t) - z\right|^2\right)\right)\right] + \mathbb{E}\left[\mathrm{tr}_N\left((g_\varepsilon(1-h))\left(\left|B_0^N B^N(t) - z\right|^2\right)\right)\right].$$

Besides, we have

$$\begin{aligned}
&\mathbb{E}\Big[\mathrm{tr}_N\left((g_\varepsilon h)\left(\left|B_0^N B^N(t) - z\right|^2\right)\right)\Big] \\
&\leq 2\mathbb{E}\left[\left\||B_0^N B^N(t) - z\right\| \mathbb{1}_{\left\|B_0^N B^N(t) - z\right\| \geq \widetilde{M}}\right] \\
&\leq 2\mathbb{E}\left[\left(\left\|B_0^N\right\|\left\|B^N(t)\right\| + |z|\right)\mathbb{1}_{\||B^N(t)\| \geq M}\right] \\
&\leq 2\left\|B_0^N\right\|\int_M^\infty \mathbb{P}\left(\left\|B^N(t)\right\| \geq s\right) ds + 2|z|\mathbb{P}\left(\left\|B^N(t)\right\| \geq M\right).
\end{aligned}$$

Thus thanks to Theorem 3.6, we get that for $M = 2c_{\varrho,t}$,

$$\begin{aligned}
\mathbb{E}\left[\mathrm{tr}_N\left((g_\varepsilon h)\left(\left|B^N(t) - z\right|^2\right)\right)\right] &\leq 2\int_{2c_{\varrho,t}}^\infty \left(\frac{c_{\varrho,t}}{s}\right)^{N^{2/3}} ds + 2|z|2^{-N^{2/3}} \\
&\leq (4\left\|B_0^N\right\| c_{\varrho,t} + 2|z|)2^{-N^{2/3}}.
\end{aligned}$$

Besides, up to a constant that only depends on t and ϱ, $\|g_\varepsilon(1-h)\|_{\mathcal{C}^5(\mathbb{R})}$ can be upper bounded by

$$(|\ln(\varepsilon)| + |\ln(|z|)|)\,\varepsilon^{-5}$$

Hence, since the support of $g_\varepsilon(1-h)$ is bounded by $2\left\|B_0^N\right\| c_{\varrho,t} + |z| + 1$, we get the conclusion thanks to Corollary 5.5. □

For the small singular values, we will proceed by first studying the mesoscopic regim.

Lemma 6.3. *There exists a constant $C = C(t,\varrho,\zeta)$ such that for any $z \in \mathbb{C}$ and $\varepsilon \geq 0$,*

$$\mathbb{E}\left[\mu_{|B^N(t)-z|}([0,\varepsilon])\right] \leq C\left(\sqrt{\varepsilon} + \frac{\ln^2(N)}{N}\right).$$

Besides by letting N go to infinity,

$$\mu_{|b(t)-z|}([0,\varepsilon]) \leq C\sqrt{\varepsilon}.$$

Proof. Thanks to Proposition 4.1, we have that

$$\begin{aligned}
&\mathbb{E}\left[\mu_{|B^N(t)-z|}([0,\varepsilon])\right] \\
&\leq \frac{\ln^2(N)}{N} + \mathbb{E}\left[\frac{\#\{l \in [\lfloor\ln^2(N)\rfloor, N-1] \mid \sigma_{N-l}(B^N(t)-z) \leq \varepsilon\}}{N}\right] \\
&\leq \frac{\ln^2(N)}{N} + \mathbb{E}\left[\frac{\#\{l \in [\lfloor\ln^2(N)\rfloor, N-1] \mid c(l/N)^2 \leq \sigma_{N-l}(B^N(t)-z) \leq \varepsilon\}}{N}\right]
\end{aligned}$$

$$
\begin{aligned}
&+ \mathcal{O}(N^2 e^{-\ln^2(N)}) + e^{-N^{2/3}} \\
&\leq \frac{\ln^2(N)}{N} + \frac{\sqrt{\varepsilon} c^{-1/2} N}{N} + \mathcal{O}(N^2 e^{-\ln^2(N)}) + e^{-N^{2/3}} \\
&\leq \mathcal{O}\left(\frac{\ln^2(N)}{N}\right) + c^{-1/2}\sqrt{\varepsilon}.
\end{aligned}
$$

Hence the conclusion. □

We can now prove the following theorem.

Theorem 6.4. *Given $f : \mathbb{C} \to \mathbb{C}$ a $\mathcal{C}^2$ function with compact support, then*

$$
\mathbb{E}\left[\mu_{B^N(t)}(f)\right] = \mu_{b(t)}(f) + \mathcal{O}\left(\frac{\|f\|_\Delta \ln(N)}{N^{\frac{2}{11}}}\right),
$$

where $\|f\|_\Delta = \int_{\mathbb{C}} |\Delta f(z)| dz$.

Proof. First, with g_ε as in Lemma 6.2, we get that for $\alpha \leq \varepsilon/2$,

$$
\begin{aligned}
\mathbb{E}\left[\mu_{B^N(t)}(f)\right] &= \mathbb{E}\left[\int_{\mathbb{C}} \Delta f(z) \operatorname{tr}_N\left(\mathbb{1}_{[0,\alpha]} \ln(|B_0^N b(t) - z|)\right) dz\right] \\
&\quad + \frac{1}{2}\mathbb{E}\left[\int_{\mathbb{C}} \Delta f(z) \operatorname{tr}_N\left(\left(\mathbb{1}_{[\alpha,\infty)} \ln - g_\varepsilon\right)(|B_0^N b(t) - z|^2)\right) dz\right] \\
&\quad + \frac{1}{2}\mathbb{E}\left[\int_{\mathbb{C}} \Delta f(z) \operatorname{tr}_N\left(g_\varepsilon(|B_0^N b(t) - z|^2)\right) dz\right].
\end{aligned}
$$

Besides, thanks to the first part of Proposition 4.1, if $\alpha \leq 1$,

$$
\begin{aligned}
&\left|\mathbb{E}\left[\int_{\mathbb{C}} \Delta f(z) \operatorname{tr}_N\left(\mathbb{1}_{[0,\alpha]} \ln(|B_0^N b(t) - z|)\right) dz\right]\right| \\
&\leq \|f\|_\Delta \sup_{z\in\mathbb{C}} \mathbb{E}\left[-\ln(\sigma_{\min}(B_0^N b(t) - z)) \mathbb{1}_{\sigma_{\min}(B_0^N b(t)-z)\leq\alpha}\right] \\
&= \|f\|_\Delta \sup_{z\in\mathbb{C}} \int_0^\infty \mathbb{P}\left(-\ln(\sigma_{\min}(B_0^N b(t) - z)) \geq \max(|\ln\alpha|, t)\right) dt \\
&= \|f\|_\Delta \sup_{z\in\mathbb{C}} \int_0^\infty \mathbb{P}\left(\sigma_{\min}(B_0^N b(t) - z) \geq e^{-\max(|\ln\alpha|,t)}\right) dt \\
&\leq CN^2 \|f\|_\Delta \int_0^\infty e^{-\max(|\ln\alpha|,t)/2} dt \\
&= CN^2 \|f\|_\Delta \sqrt{\alpha}(2 + |\ln\alpha|).
\end{aligned}
$$

We also have thanks to Lemma 6.3 that,

$$
\begin{aligned}
&\frac{1}{2}\mathbb{E}\left[\int_{\mathbb{C}} \Delta f(z) \operatorname{tr}_N\left(\left(\mathbb{1}_{[\alpha,\infty)} \ln - g_\varepsilon\right)(|B_0^N b(t) - z|^2)\right) dz\right] \\
&\leq \|f\|_\Delta |\ln(\alpha)| \sup_{z\in\mathbb{C}} \mathbb{E}\left[\mu_{|B^N(t)-z|}([0,\varepsilon])\right] \\
&\leq C \|f\|_\Delta |\ln(\alpha)| \left(\sqrt{\varepsilon} + \frac{\ln^2(N)}{N}\right).
\end{aligned}
$$

And thanks to Lemma 6.2, we get that

$$
\left|\frac{1}{2}\mathbb{E}\left[\int_{\mathbb{C}} \Delta f(z) \operatorname{tr}_N\left(g_\varepsilon\left(\left|B^N(t) - z\right|^2\right)\right) dz\right] - \frac{1}{2}\int_{\mathbb{C}} \Delta f(z) \tau\left(g_\varepsilon\left(|b(t) - z|^2\right)\right) dz\right|
$$

$$\leq C\frac{|\ln(\varepsilon)|}{\varepsilon^5 N^2}\|f\|_\Delta .$$

Finally, thanks to Lemma 6.3, we have that

$$\begin{aligned}
&\left|\int_{\mathbb{C}} \Delta f(z)\tau\left(\ln\left(|b(t)-z|\right)\right)dz - \frac{1}{2}\int_{\mathbb{C}}\Delta f(z)\tau\left(g_\varepsilon\left(|b(t)-z|^2\right)\right)dz\right| \\
&\leq \|\Delta f\|\sup_{z\in\mathbb{C}}\tau\left(-\ln\left(|b(t)-z|\right)\mathbb{1}_{[0,\varepsilon]}(|b(t)-z|\right) \\
&= \|\Delta f\|\sup_{z\in\mathbb{C}}\int_0^\infty \mu_{|b(t)-z|}\left([0,\min(e^{-t},\varepsilon)]\right)dt \\
&\leq C\,\|\Delta f\|\,\sqrt{\varepsilon}(|\ln\varepsilon|+2).
\end{aligned}$$

Thus we get that

$$\begin{aligned}
&\mathbb{E}\left[\mu_{B^N(t)}(f)\right] - \mu_{b(t)}(f) \\
&= \mathcal{O}\left(\|f\|_\Delta\left(N^2\sqrt{\alpha}|\ln\alpha| + |\ln(\alpha)|\left(\sqrt{\varepsilon}+\frac{\ln^2(N)}{N}\right) + \frac{|\ln(\varepsilon)|}{\varepsilon^5N^2} + \sqrt{\varepsilon}|\ln\varepsilon|\right)\right).
\end{aligned}$$

Hence the conclusion by taking $\alpha = N^{-8}$ and $\varepsilon = N^{-4/11}$. □

6.3. **Convergence of the spectrum.** The aim of this subsection is to prove the following theorem.

Theorem 6.5. *Given $B^N(t)$ a Brownian motion on $\mathrm{GL}(N,\mathbb{C})$ at time t, then almost surely the spectrum of $B^N(t)$ converges for the Hausdorff distance towards the support Σ_t of the Brown measure of the free multiplicative motion $b(t)$.*

We know that almost surely the empirical law $\mu_{B^N(t)}$ of eigenvalues of $B^N(t)$ converges towards the Brown measure μ_t of $b(t)$. Besides, since Σ_t is compact, let $(z_i)_{1\leq i\leq n}$ be such that $\Sigma_t \subset \cup_{i=1}^n B(z_i,\varepsilon/2)$ (where $B(z,\varepsilon)$ is the open ball in $\mathbb{C}$ centered in z and of radius ε), we also define $f_i : \mathbb{C}\to\mathbb{C}$ a function which is positive on $B(z_i,\varepsilon/2)$ and equal to 0 elsewhere. We know that almost surely, for every i, $\mu_{B^N(t)}(f_i)$ converges towards $\mu_t(f_i) > 0$, hence for N large enough, for all i, $\mu_{B^N(t)}(f_i) > 0$. However $\mu_{B^N(t)}(f_i)$ is positive if and only if there is an element of the spectrum of $B^N(t)$ in $B(z_i,\varepsilon/2)$. Consequently, almost surely, for N large enough,

$$\Sigma_t \subset \sigma(B^N(t)) + \varepsilon := \{\lambda\in\mathbb{C} \mid \exists z\in\sigma(B^N(t)),\ |z-\lambda|<\varepsilon\}.$$

The other inclusion is a consequence of the fact that $\Sigma_t = \sigma(b(t))$, and that the Brownian motion on $\mathrm{GL}(N,\mathbb{C})$ converges strongly towards $b(t)$ (see [11]). Indeed we can then use the following lemma.

Todd could you add the reference here?

Lemma 6.6. *Given X^N a matrix which converges strongly in $*$-distribution towards x (see Definition 2.4), then for all $\varepsilon>0$, for N large enough,*

$$\sigma(X^N) \subset \sigma(x) + \varepsilon := \{\lambda\in\mathbb{C} \mid \exists z\in\sigma(x),\ |z-\lambda|<\varepsilon\}.$$

Proof. To begin with, by strong convergence of X^N, we know that

$$R := \sup_N \left\|X^N\right\|$$

is finite. Thus the spectrum of X^N is in a ball of radius R centered in 0. Let $\lambda\in B(0,R)\setminus(\sigma(x)+\varepsilon)$, then λ is not in the spectrum of x, hence $x-\lambda$ is invertible, and thus so is $|x-\lambda|^2$. In particular, since the spectrum of $|x-\lambda|^2$ is compact, it

is disjoint of $[-c_\lambda, c_\lambda]$ for some constant $c_\lambda > 0$. Consequently, by using the strong convergence of X^N, we have that the spectrum of $|X^N - \lambda|^2$ converges towards the one of $|x-\lambda|^2$. And thus, for N large enough, the spectrum of $|X^N - \lambda|^2$ is disjoint of $[-c_\lambda/2, c_\lambda/2]$.

Since almost surely, for N large enough, $\left\|X^N\right\| \leq R$, we have that there exists a constant κ_λ such that for N large enough, for all $z \in B(\lambda, \kappa_\lambda)$, $|X^N - z|^2$ (and thus also $B^N(t) - z$) is invertible. By compacity of $B(0,R) \setminus (\Sigma_t + \varepsilon)$, one can cover this set by a finite number of balls of the form $B(\lambda, \kappa_\lambda)$, hence, for N large enough, $X^N - z$ is invertible for any $z \in B(0,R) \setminus (\Sigma_t + \varepsilon)$. In particular since we already know that $\sigma(X^N) \subset B(0,R)$, we get that for N large enough, $\sigma(X^N) \subset \sigma(x) + \varepsilon$. Hence the conclusion.

□

7. The self-adjoint case

We want to show that there exist constants c and C such that for X_N a GUE random matrix, any deterministic matrices A_N,

$$\mathbb{P}(\sigma_{\min}(X_N + A_N) \leq \varepsilon) \leq \varepsilon^c N^C.$$

Note that if A_i is the event

$$\left\{ \inf_{\|x\|_2 = 1} \|(X_N + A_N)x\|_2 \leq \varepsilon, \text{ the infimum is reached at } x \text{ and } |x_i| \geq \frac{1}{\sqrt{N}} \right\},$$

then

$$\mathbb{P}(\sigma_{\min}(X_N + A_N) \leq \varepsilon) \leq \sum_{i=1}^{N} \mathbb{P}\left(A_i\right).$$

Besides, for any unitary matrix U, $\inf_{\|x\|_2=1} \|(X_N + A_N)x\|_2$ has the same law as

$$\inf_{\|x\|_2=1} \|(U^* X_N U + A_N)x\|_2 = \inf_{\|x\|_2=1} \|(X_N + U A_N U^*)x\|_2 .$$

Consequently,

$$\mathbb{P}(\sigma_{\min}(X_N + A_N) \leq \varepsilon) \leq N \sup_{A^N \in \mathbb{M}_N(\mathbb{C})} \mathbb{P}\left(\inf_{\|x\|_2=1,\ |x_1| \geq \frac{1}{\sqrt{N}}} \|(X_N + A_N)x\|_2 \leq \varepsilon \right).$$

Thanks to [78], Lemma 3.2, if we write

$$X_N + A_N = \begin{bmatrix} A_{1,1} + X_{1,1} & (X + A)^T \\ (X^* + \widetilde{A})^T & B^T \end{bmatrix},$$

then

$$\inf_{\|x\|_2=1,\ |x_1| \geq \frac{1}{\sqrt{N}}} \|(X_N + A_N)x\|_2 \geq \frac{1}{\sqrt{N}} \frac{|A_{1,1} + X_{1,1} - (X^* + \widetilde{A})B^{-1}(X + A)|}{\sqrt{1 + \|B^{-1}(X + A)\|_2^2}}.$$

However, with g a centered Gaussian random variable of variance 1, one has that

$$\begin{aligned} &\mathbb{P}\left(\frac{|A_{1,1} + X_{1,1} - (X^* + \widetilde{A})B^{-1}(X + A)|}{\sqrt{1 + \|B^{-1}(X + A)\|_2^2}} \leq \varepsilon\sqrt{N},\ \left\|B^{-1}(X + A)\right\|_2 \leq K \right) \\ &\leq \mathbb{P}\left(|g + Z| \leq \varepsilon(K+1)N\right) \\ &\leq \varepsilon(K+1)N, \end{aligned}$$

where Z is a random variable independent of g.

Besides, if $\left\|B^{-1}\right\| \geq L$, then there exists $\omega \in \mathbb{C}^{N-1}$ of norm 1 such that $\left\|B^{-1}\omega\right\|_2 \geq L$, hence with $y = \frac{B^{-1}\omega}{\|B^{-1}\omega\|_2}$, $\|By\|_2 \leq \frac{1}{L}$, hence if $|x_1| \geq N^{-1/2}$,

$$\begin{aligned}\|(X_N + A_N)x\|_2^2 &\geq \left|(0\ y^T)\begin{bmatrix} A_{1,1} + X_{1,1} & (X+A)^T \\ (X^* + \widetilde{A})^T & B^T \end{bmatrix}\begin{bmatrix} x_1 \\ \widetilde{x}\end{bmatrix}\right| \\ &\geq |x_1 y^T (X^* + \widetilde{A})^T + y^T B^T \widetilde{x}| \\ &\geq \frac{1}{\sqrt{N}}|X^* y + \widetilde{A} y| - \|By\|_2 \\ &\geq \frac{1}{\sqrt{N}}|X^* y + \widetilde{A} y| - \frac{1}{L}.\end{aligned}$$

Since y has norm 1 and is independent of X^*, X^*y has the same law as $\frac{x+\mathbf{i}y}{\sqrt{2N}}$ where x and y are independent Gaussian random variables of variance 1, thus

$$\begin{aligned}&\mathbb{P}\left(\inf_{\|x\|_2=1,\ |x_1|\geq\frac{1}{\sqrt{N}}} \|(X_N + A_N)x\|_2 \leq \varepsilon, \left\|B^{-1}\right\| \geq L\right) \\ &\leq \mathbb{P}\left(|g + Z| \leq \sqrt{2}N\left(\varepsilon^2 + \frac{1}{L}\right)\right) \\ &\leq \sqrt{2}N\left(\varepsilon^2 + \frac{1}{L}\right).\end{aligned}$$

where g a centered Gaussian random variable of variance 1, Z is a random variable independent of g.

Besides, by Markov inequality

$$\mathbb{P}\left(\|X\|_2 \geq S\right) \leq \frac{\mathbb{E}\left[\|X\|_2^2\right]}{S^2} = \frac{N-1}{NS^2} \leq \frac{1}{S^2}.$$

Thus we are left with

$$\mathbb{P}\Bigg(\frac{|A_{1,1} + X_{1,1} - (X^* + \widetilde{A})B^{-1}(X+A)|}{\sqrt{1 + \|B^{-1}(X+A)\|_2^2}} \leq \varepsilon\sqrt{N}, \\ \left\|B^{-1}(X+A)\right\|_2 \geq K,\ \left\|B^{-1}\right\| \leq L,\ \|X\|_2 \leq S\Bigg).$$

However if $\left\|B^{-1}(X+A)\right\|_2 \geq K,\ \left\|B^{-1}\right\| \leq L,\ \|X\|_2 \leq S$, then

$$\begin{aligned}&\frac{|A_{1,1} + X_{1,1} - (X^* + \widetilde{A})B^{-1}(X+A)|}{\sqrt{1 + \|B^{-1}(X+A)\|_2^2}} \\ &\geq \frac{|A_{1,1} + X_{1,1} - X^*B^{-1}A - \widetilde{A}B^{-1}X - \widetilde{A}B^{-1}A| - LS^2}{\|B^{-1}A\|_2\left(1 + \frac{LS}{K - LS}\right)}.\end{aligned}$$

If $X = \frac{1}{\sqrt{2N}}(U + \mathbf{i}V)$, where U and V are independent real Gaussian vector, then with $R := A_{1,1} + X_{1,1} - \widetilde{A}B^{-1}A$,

$$\frac{|A_{1,1} + X_{1,1} - (X^* + \widetilde{A})B^{-1}(X + A)|}{\sqrt{1 + \|B^{-1}(X + A)\|_2^2}}$$
$$\geq \frac{|R - \frac{1}{\sqrt{2N}}U^T(B^{-1}A + (\widetilde{A}B^{-1})^T) + \frac{\mathbf{i}}{\sqrt{2N}}V^T(B^{-1}A - (\widetilde{A}B^{-1})^T)| - LS^2}{\|B^{-1}A\|_2\left(1 + \frac{LS}{K - LS}\right)}.$$

And since

$$\left\|B^{-1}A\right\| \leq \left\|B^{-1}A - (\widetilde{A}B^{-1})^T\right\|_2 + \left\|B^{-1}A + (\widetilde{A}B^{-1})^T\right\|_2,$$

we have that either

$$\frac{\left\|B^{-1}A - (\widetilde{A}B^{-1})^T\right\|_2}{\|B^{-1}A\|_2} \geq \frac{1}{2}, \quad \text{or} \quad \frac{\left\|B^{-1}A + (\widetilde{A}B^{-1})^T\right\|_2}{\|B^{-1}A\|_2} \geq \frac{1}{2},$$

In particular there is one entry of one of those vectors whose modulus is at least $\frac{1}{\sqrt{2N}}$, thus

$$\frac{|A_{1,1} + X_{1,1} - (X^* + \widetilde{A})B^{-1}(X + A)|}{\sqrt{1 + \|B^{-1}(X + A)\|_2^2}} \geq \frac{1}{2N}\frac{|g + Z| - \frac{2NLS^2}{K - LS}}{\left(1 + \frac{LS}{K - LS}\right)},$$

where g is a centered Gaussian random variable of variance 1, Z a random variable independent of g. Consequently,

$$\mathbb{P}\Bigg(\frac{|A_{1,1} + X_{1,1} - (X^* + \widetilde{A})B^{-1}(X + A)|}{\sqrt{1 + \|B^{-1}(X + A)\|_2^2}} \leq \varepsilon\sqrt{N},$$
$$\left\|B^{-1}(X + A)\right\|_2 \geq K,\ \left\|B^{-1}\right\| \leq L,\ \|X\|_2 \leq S\Bigg)$$
$$\leq 2N\mathbb{P}\left(|g + Z| \leq 2N^{3/2}\varepsilon\left(1 + \frac{LS}{K - LS}\right) + \frac{2NLS^2}{K - LS}\right)$$
$$\leq 4N^2\left(\sqrt{N}\varepsilon\left(1 + \frac{LS}{K - LS}\right) + \frac{LS^2}{K - LS}\right)$$

Consequently, we get that

$$\mathbb{P}(\sigma_{\min}(X_N + A_N) \leq \varepsilon) \leq \varepsilon(K + 1)N^2 + \sqrt{2}N^2\left(\varepsilon^2 + \frac{1}{L}\right) + \frac{N}{S^2}$$
$$+ 4N^3\left(\sqrt{N}\varepsilon\left(1 + \frac{LS}{K - LS}\right) + \frac{LS^2}{K - LS}\right).$$

Hence, if we take $L = \varepsilon^{-1/4}$, $S = \varepsilon^{-1/8}$ and $K = 2\varepsilon^{-3/4}$,

$$\mathbb{P}(\sigma_{\min}(X_N + A_N) \leq \varepsilon) \leq 3\varepsilon^{1/4}N^2 + 2\sqrt{2}N^2\varepsilon^{1/4} + N\varepsilon^{1/4}$$
$$+ 4N^3\left(\sqrt{N}\varepsilon\left(1 + \varepsilon^{1/4}\right) + \varepsilon^{1/4}\right)$$
$$\leq 20N^{7/2}\varepsilon^{1/4}.$$

Hence the conclusion.

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