HYPERCONTRACTIVITY FOR LOG-SUBHARMONIC FUNCTIONS

PIOTR GRACZYK(1), TODD KEMP(2), AND JEAN-JACQUES LOEB(1)

ABSTRACT. We prove strong hypercontractivity (SHC) inequalities for logarithmically subharmonic functions on $\mathbb{R}^n$ and different classes of measures: Gaussian measures on $\mathbb{R}^n$, symmetric Bernoulli and symmetric uniform probability measures on $\mathbb{R}$, as well as their convolutions. Surprisingly, a slightly weaker strong hypercontractivity property holds for *any* symmetric measure on $\mathbb{R}$. A log–Sobolev inequality (LSI) is deduced from the (SHC) for compactly supported measures on $\mathbb{R}^n$, still for log–subharmonic functions. An analogous (LSI) is proved for Gaussian measures on $\mathbb{R}^n$ and for other measures for which we know the (SHC) holds. Our log–Sobolev inequality holds in the log-subharmonic category with a constant *smaller* than the one for Gaussian measure in the classical context.

1. INTRODUCTION

In this paper, we prove some important inequalities – strong hypercontractivity (SHC) and a logarithmic Sobolev inequality – for logarithmically subharmonic functions (cf. Definition 2.1 below). Our paper is inspired by work of Janson [15], in which he began the study of an important property of semigroups called **strong hypercontractivity**. A rich series of subsequent papers by Janson [16], Carlen [4], Zhao [20], and recently by Gross ([10, 11] and a survey [12]) was devoted to this subject on the spaces $\mathbb{C}^n$ and, in papers by Gross, on complex manifolds. In contrast to all the aforementionned papers, our results concern the real spaces $\mathbb{R}^n$.

In the first part of the paper (Sections 3–4) we prove strong hypercontractivity in the log–subharmonic setting: for $0 < p \le q < \infty$,

$$\|T_t f\|_{L^q(\mu)} \le \|f\|_{L^p(\mu)} \quad \text{for} \quad t \ge \frac{1}{2}\log\frac{q}{p} \tag{SHC}$$

for the dilation semigroup $T_t f(x) = f(e^{-t}x)$, for any logarithmically subharmonic function f, for different classes of measures μ: including Gaussian measures and some compactly supported measures on $\mathbb{R}$ (symmetric Bernoulli and uniform probability measure on $[-a, a]$ for $a > 0$). We also show that, in numerous important cases, the convolution of two measures satisfying (SHC) also satisfies (SHC).

Let us note that in the theory of hypercontractivity for general measures, the semigroup considered is the one associated to the measure by the usual technology of Dirichlet forms. The generator of the semigroup (on a complete Riemannian manifold) takes the form $-\Delta + X$ where Δ is the Laplace-Beltrami operator and X is a vector field; hence, the semigroup restricted to *harmonic functions* on the manifold is simply the (backward) flow of X. For Gaussian measure, $X = x \cdot \nabla$, yielding the above flow T_t; this vector field is often called the *Euler operator*, denoted E. In a sense, the point of this paper is to show that the strong hypercontractivity theorems about this flow extend beyond harmonic functions to the larger class of logarithmically subharmonic functions.

The second part of the paper (Section 5) is devoted to Logarithmic Sobolev Inequalities (LSI) corresponding to the Strong Hypercontractivity property for log–subharmonic functions. We prove a general implication (SHC) $\Rightarrow$ (LSI) for compactly supported measures on $\mathbb{R}^n$ for log–subharmonic functions. (It is important to note that, while the general technique of this implication – differentiating the inequalities in an appropriate fashion – are well-known, the technical details here involved with regularizing subharmonic functions are quite difficult.) We also show that an analogous log–Sobolev inequality in the log–subharmonic domain holds for Gaussian measures on $\mathbb{R}^n$

Date: May 5, 2009.
(2) This work was partially supported by NSF Grant DMS-0701162.

and for other measures which satisfy the strong hypercontractivity (SHC) considered in the first part. In both cases, the (LSI) we get is *stronger* than the classical one in the following sense. Let

$$t_N(p,q) = \frac{1}{2}\log\frac{q-1}{p-1}, \quad t_J(p,q) = \frac{1}{2}\log\frac{q}{p}$$

denote the Nelson and Janson times (cf. [18, 15]), for $1 < p \le q < \infty$ (in fact, t_J makes sense for all positive $p \le q$). The classical hypercontractivity for $t \ge c\,t_N$ is equivalent, by Gross's theorem in [9], to a logarithmic-Sobolev inequality with the constant $2c$:

$$\int |f|^2 \log|f|^2 d\mu - \|f\|_{2,\mu}^2 \log\|f\|_{2,\mu}^2 \le 2c\int fLf d\mu$$

where L is the positive generator of the semigroup. We show that, in the category of logarithmically subharmonic functions, the strong hypercontractivity for $t \ge c\,t_J$ implies (LSI) with *constant* c:

$$\int |f|^2 \log|f|^2 d\mu - \|f\|_{2,\mu}^2 \log\|f\|_{2,\mu}^2 \le c\int fEf d\mu \tag{LSI}$$

where E is the Euler operator discussed above. Hence, one cannot obtain this stronger LSI by simply restricting the usual Gaussian LSI to log–subharmonic functions. We call the inequality (LSI) a "**strong LSI**" both because it corresponds to the strong hypercontractivity and as the constant in the energy integral is smaller than in the classical case (of the Gaussian LSI of [9]). (LSI) could also be appropriately called an *Euler type* Logarithmic Sobolev Inequality.

We emphasize the fact that the strong (LSI) and the implication (SHC) $\Rightarrow$ strong (LSI) were never observed before in holomorphic case, in the afore–mentionned papers on strong hypercontractivity. In [10], only the implication classical (LSI)$\Rightarrow$ (SHC) is proved. The authors of [13] observe and extensively discuss the difficulty in approximating of subharmonic functions. Let us note that the implication (SHC) $\Rightarrow$ (LSI) in the log–subharmonic case does not follow as easily as in the classical setting. Indeed, if f is log–subharmonic, the functions $f|_{[-N,N]}$ and $f\mathbf{1}_{|f|<N}$ are not log–subharmonic on $\mathbb{R}$, and the classical techniques of approximation by more regular (e.g. compactly supported or bounded) functions fail.

Let us mention that some interesting Log-Sobolev type inequalities were proved for log–convex functions and a large class of measures in [1]. Those inequalities are essentially different from ours, whose right-hand side comes from the Dirichlet form of the semigroup, like in the classical LSI.

Our principal reference for the basic preliminaries is the book [2] which gives a very accessible survey on hypercontractivity and on logarithmic–Sobolev inequalities.

Acknowledgment. We thank A. Hulanicki for calling the attention of the first and third authors to hypercontractivity problems in the holomorphic category. Thanks also go to L. Gross for many helpful conversations and to an anonimous referee for numerous improvements of the paper.

2. LOG–SUBHARMONIC FUNCTIONS

Definition 2.1. An $L^1_{\rm loc}$ upper semi-continuous function $f : \mathbb{R}^n \to [-\infty, +\infty)$, not identically equal to $-\infty$, is called **subharmonic** if for every $x, y \in \mathbb{R}^n$, one has the inequality:

$$f(x) \le \fint_{O(n)} f(x+\alpha y)\, d\alpha \tag{2.1}$$

where $O(n)$ is the orthogonal group of $\mathbb{R}^n$ and $d\alpha$ is the normalized Haar measure on it. (The notation $\fint$ is a reminder that the measure in question is normalized.) When $f \in \mathcal{C}^2$ then the Definition 2.1 is equivalent to $\Delta f \ge 0$. Let us also recall that the subharmonic functions satisfy the maximum principle.

A non-negative function $g : \mathbb{R}^n \to [0, +\infty)$ is called **log–subharmonic** (abbreviated LSH) if the function $\log g$ is subharmonic.

Remark 2.1. Definition 2.1 is evidently equivalent to insisting that $f(x) \leq \fint_{\partial B(x,r)} f(t)\,\sigma(dt)$ for every $x \in \mathbb{R}^n$, where $\partial B(x,r)$ is the sphere of radius r about the point x, and σ is normalized Lebesgue measure on this sphere. Frequently, subharmonicity is stated in terms of averages over solid balls $B(x,r)$ instead; the two approaches are equivalent for L^1_{loc} upper-semicontinuous functions. Subharmonic function (and ergo log-subharmonic functions) need not have very good local properties. There are subharmonic functions that are discontinuous everywhere (see, for example, [19]). In some of what follows, it will be convenient to work with *continuous* LSH functions; where this restriction is in place, we have stated it explicitly.

Example 2.1. The following examples of LSH functions are well-known and easily verified.

(1) A convex function is subharmonic. On $\mathbb{R}$, f is subharmonic if and only if f is convex.
(2) Let f be a holomorphic function on $\mathbb{C}^n$. Then $|f|$ is a log–subharmonic function (see [14] or use Jensen's inequality). Indeed, $\log|f|$ is actually harmonic on the complement of $\{f = 0\}$.
(3) Denote by $\langle\ ,\ \rangle$ the scalar product on $\mathbb{R}^n$, and fix $a \in \mathbb{R}^n$. Then $x \mapsto \exp\langle a, x\rangle$ is a log–subharmonic function.

The main content of the next proposition is item 2, which takes some work to prove and will be important in what follows.

Proposition 2.2. *Let f, g be* LSH*, and let $p > 0$.*

(1) *The product fg is* LSH*, as is g^p.*
(2) *The sum $f + g$ is* LSH.
(3) *f is subharmonic.*

Proof. Property 1 is evident. In order to prove 3 (note that non-negativity is built into the definition of LSH functions), we use the fact that if a function $\varphi : \mathbb{R} \to \mathbb{R}$ is increasing and convex and h is a subharmonic function then $\varphi(h)$ is also subharmonic. We apply this fact with $\varphi(x) = e^x$ and $h = \log f$ when f is LSH. To prove 2, we need the following lemma.

Lemma 2.3. *Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a convex function of two variables, increasing in each variable. If F and G are subharmonic functions then $\varphi(F, G)$ is also subharmonic.*

Proof. We apply the Jensen inequality in dimension 2

$$\varphi(F(x), G(x)) \leq \varphi\left(\int_{O(n)} F(x + \alpha y)d\alpha, \int_{O(n)} G(x + \alpha y)d\alpha\right) \leq \int \varphi(F, G)(x + \alpha y)d\alpha.$$

□

It is easy to verify that the function $\varphi(x, y) = \log(e^x + e^y)$ satisfies the hypotheses of the lemma: to check its convexity, we write $\log(e^x + e^y) = x + \log(1 + e^{x-y})$, yielding the result since the function $t \mapsto \ln(1 + e^t)$ is convex. Hence, if f and g are LSH, then $f = e^F$ and $g = e^G$ for subharmonic functions F, G, and so the lemma yields that $\varphi(F, G) = \log(f + g)$ is subharmonic. This ends the proof of the proposition. □

The next lemma and corollary are based on Proposition 2.2. They are useful in much of the following.

Lemma 2.4. *Let Ω be a separable metric space, and let μ a Borel probability measure on Ω. Suppose $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ satisfies*

(1) *The function $x \mapsto f(\omega, x)$ is* LSH *and continuous for μ–almost every $\omega \in \Omega$.*
(2) *The function $\omega \mapsto f(\omega, x)$ is bounded and continuous for each $x \in \mathbb{R}^n$.*
(3) *For small $r > 0$, there is a constant $C_r > 0$ so that, for all $\omega \in \Omega$ and all $x \in \mathbb{R}^n$, $|f(\omega, t)| \leq C_r$ for $t \in B(x, r)$.*

Then the function $\tilde{f}(x) = \int_\Omega f(\omega, x)\,\mu(d\omega)$ is LSH.

Proof. By Varadarajan's theorem (see Theorem 11.4.1 in [5]), there is a sequence of points $\omega_j \in \Omega$ such that the probability measures

$$\mu_n = \frac{1}{n}\sum_{j=1}^{n} \delta_{\omega_j}$$

converge weakly to μ: $\mu_n \rightharpoonup \mu$. Note that

$$\tilde{f}_n(x) = \int_{\Omega} f(\omega, x)\mu_n(d\omega) = \frac{1}{n}\sum_{j=1}^{n} f(\omega_j, x),$$

and by Proposition 2.2 part (2), $\tilde{f}_n$ is LSH for each n. Moreover, since $f(\,\cdot\,, x) \in C_b(\Omega)$, weak convergence guarantees that $\tilde{f}_n(x) \to \tilde{f}(x)$ for each x. Fix $\epsilon > 0$; then since $\tilde{f}_n$ and $\tilde{f}$ are non-negative, $\tilde{f}_n + \epsilon$ and $\tilde{f} + \epsilon$ are strictly positive and thus $\log(\tilde{f}_n(x) + \epsilon) \to \log(\tilde{f}(x) + \epsilon)$ for each x. Again using Proposition 2.2, $\tilde{f}_n + \epsilon$ is LSH and so $\log(\tilde{f}_n + \epsilon)$ is subharmonic. Let $r > 0$ be small, and consider

$$\fint_{\partial B(x,r)} \log(\tilde{f}(t) + \epsilon)\, dt = \fint_{\partial B(x,r)} \lim_{n\to\infty} \log(\tilde{f}_n(t) + \epsilon)\, dt.$$

By assumption, $|f(\omega, t)| \le C_r$ for each $\omega \in \Omega$ and $t \in \partial B(x, r)$; hence, $|\tilde{f}_n(t)| \le C_r$ a well. This means there is a uniform bound on $\log(\tilde{f}_n + \epsilon)$ on $\partial B(x, r)$. We may therefore apply the dominated convergence theorem to find that

$$\begin{aligned}\fint_{\partial B(x,r)} \log(\tilde{f}(t) + \epsilon)\, dt &= \lim_{n\to\infty} \fint_{\partial B(x,r)} \log(\tilde{f}_n(t) + \epsilon)\, dt \\ &\ge \lim_{n\to\infty} \log(\tilde{f}_n(x) + \epsilon) = \log(\tilde{f}(x) + \epsilon),\end{aligned}$$

where the inequality follows from the fact that $\log(\tilde{f}_n + \epsilon)$ is subharmonic. Hence, $\tilde{f} + \epsilon$ is LSH for each $\epsilon > 0$. Finally, since $f(\omega, x)$ is continuous in x for almost every ω, the boundedness of f in ω shows that $\tilde{f}$ is continuous. Thus the set where $\tilde{f} > -\infty$ is open. Therefore $\log(\tilde{f}(x) + \epsilon)$ is uniformly-bounded in ϵ on small enough balls around x, and a simple argument like the one above shows that the limit as $\epsilon \downarrow 0$ can be performed to show that $\tilde{f}$ is LSH as required. □

Remark 2.5. It is possible to dispense with the requirement that $f(\omega, x)$ is continuous in x by using Fatou's lemma instead of the dominated convergence theorem; however, the continuity of $f(\omega, x)$ in ω is still required for this argument. In all the applications we have planned for Corollary 2.4, $f(\omega, x)$ is such that continuity in one variable implies continuity in the other, and so we need not work harder to eliminate this hypothesis.

Remark 2.6. In Lemma 2.4, if LSH is replaced with the weaker condition *lower-bounded subharmonic* (in the premise and conclusion of the statement), then the result follows from Definition 2.1 with a simple application of Fubini's theorem; moreover, the only assumption needed is that $f(\,\cdot\,, x) \in L^1(\Omega, \mu)$ for each x.

Corollary 2.7. *Suppose $f\colon \mathbb{R}^n \to \mathbb{R}$ is lower-bounded and subharmonic. Then the function*

$$\tilde{f}(x) = \int_{O(n)} f(\alpha x)\, d\alpha$$

is subharmonic. Moreover, if f is also LSH *and continuous, then so is $\tilde{f}$. In either case, $\tilde{f}$ depends only on the radial direction: there is a function $g\colon [0,\infty) \to [-\infty, \infty)$ with $\tilde{f}(x) = g(|x|)$, and g is* non-decreasing *on* $[0, \infty)$.

Proof. Suppose f is LSH and continuous. The reader may readily verify that the function $(\alpha, x) \mapsto f(\alpha x)$ satisfies all the conditions of Lemma 2.4. (The weaker statement for lower-bounded subharmonic f, not necessarily continuous, follows similarly via Remark 2.6.) Clearly averaging f over rotations makes $\tilde{f}$ radially symmetric. Any radially symmetric subharmonic function is radially non-decreasing, by the maximum principle. □

3. Hypercontractivity inequalities for the Gaussian measure

Let m be a probability measure on $\mathbb{R}^n$. For $p \geq 1$, we denote the norm on $L^p(m)$ by $\| \quad \|_{p,m}$. We will denote by $L^p_{\mathrm{LSH}}(m)$ the cone of log-subharmonic functions in $L^p(m)$. Let γ be the standard Gaussian measure on $\mathbb{R}^n$, i.e. $\gamma(dx) = c_n \exp(-|x|^2/2)\, dx$, where dx is Lebesgue measure and $c_n = (2\pi)^{-n/2}$.

Given a function f on $\mathbb{R}^n$, and $r \in [0,1]$, we denote by f_r the function $x \mapsto f(rx)$. The family of operators $S_r f = f_r, r \in [0,1]$ is a multiplicative semigroup, whose additive form $T_t f(x) = f(e^{-t}x)$ is considered in connection with holomorphic function spaces in [4, 10, 15, 20] and others (including the second author's paper [17] in the non-commutative holomorphic category). When f is differentiable, the infinitesimal generator E of $(T_t)_{t\geq 0}$ equals $-Ef$ where E is the Euler operator

$$Ef(x) = x \cdot \nabla f.$$

If L is the Ornstein–Uhlenbeck operator $L = -\Delta + E$ acting in $L^2(\mathbb{C}^n, \gamma)$ and f is a holomorphic function then $Lf = Ef$, so $(T_t)_{t\geq 0}$ and, equivalently, $(S_r)_{r\in[0,1]}$ act on holomorphic functions as the Ornstein–Uhlenbeck semigroup e^{-tL} (cf. [2] p.22–23).

Before showing the strong hypercontractivity of the semigroup S_r for the Gaussian measure and LSH functions, let us show that the operators S_r are L^p–contractions on non-negative subharmonic functions, for any rotationally invariant probability measure.

Proposition 3.1. *Let m be a probability measure on $\mathbb{R}^n$ which is $O(n)$-invariant. Then for $f \geq 0$ subharmonic , $r \in [0,1]$, and $p \geq 1$, we have*

$$\|f_r\|_{p,m} \leq \|f\|_{p,m}.$$

Moreover, this contraction property holds additionally in the regime $0 < p < 1$ if f is LSH.

Proof. First consider the case $p \geq 1$, and assume only that $f \geq 0$ is subharmonic. Note that, since $f \geq 0$ and since m is $O(n)$-invariant,

$$\|f_r\|^p_{p,m} = \int_{\mathbb{R}^n} f(rx)^p \, dm(x) = \int_{O(n)} \int_{\mathbb{R}^n} f(rx)^p \, dm(\alpha x) \, d\alpha.$$

Changing variables using the linear transformation α in the inside integral and using Fubini's theorem, we have (replacing α^{-1} with α in the end)

$$\int_{\mathbb{R}^n} \int_{O(n)} f(r\alpha x)^p \, d\alpha \, dm(x) = \int_{\mathbb{R}^n} S_r \, h(x) \, dm(x),$$

where $h(x) = \int_{O(n)} f(\alpha x)^p \, d\alpha$; i.e., with $k = f^p$, $h = \tilde{k}$ in the notation of Corollary 2.7. Since $p \geq 1$, k is subharmonic, and so by Corollary 2.7 h is also subharmonic and radially increasing. In particular, there is some non-decreasing $g\colon [0,\infty) \to \mathbb{R}$ such that $h(x) = g(|x|)$. So $S_r \, h(x) = g(r|x|) \leq g(|x|) = h(x)$ for $r \in [0,1]$. Integrating over $\mathbb{R}^n$ we have $\|f_r\|^p_{p,m} \leq \int h(x)\, dx$ which equals $\|f\|^p_{p,m}$ by reversing the above argument. This proves the result.

If $0 < p < 1$, the above argument follows through as well since, if $f \in$ LSH then $k = f^p$ is LSH by Proposition 2.2. In particular, k is non-negative and subharmonic, and so by Corollary 2.7, so is $\tilde{k}$. The rest of the proof follows verbatim.

□

We now show the strong hypercontractivity inequality for Gaussian measure and LSH functions. That is: $\|T_t f\|_{q,\gamma} \leq \|f\|_{p,\gamma}$ whenever f is LSH and $t \geq t_J(p,q)$. This is a generalization (from holomorphic functions to the much larger class of logarithmically-subharmonic functions) of Janson's original strong hypercontractivity theorem in [15]. Because our test functions f are non-negative and the action of T_t commutes with taking powers of f, this can be reduced to the following simplified form.

Theorem 3.2. *Let f be a log–subharmonic function. Then for every $r \in [0,1]$, one has*

$$\|f_r\|_{1/r^2,\gamma} \leq \|f\|_{1,\gamma}. \tag{3.1}$$

Remark 3.3. The inequality (3.1) means that the operators S_r act as contractions between the spaces

$$S_r : L^1_{\rm LSH}(\gamma) \to L^{1/r^2}_{\rm LSH}(\gamma),$$

or, equivalently, the operator T_t is a contraction between the cones

$$T_t : L^1_{\rm LSH}(\gamma) \to L^{e^{2t}}_{\rm LSH}(\gamma).$$

In fact, by Proposition 2.2, one gets other hypercontractivity properties. Applying the theorem to the function f^p, it follows that the operators S_r are contractions

$$S_r : L^p_{\rm LSH}(\gamma) \to L^{p/r^2}_{\rm LSH}(\gamma),$$

and the operators T_t are contractions

$$T_t : L^p_{\rm LSH}(\gamma) \to L^{e^{2t}p}_{\rm LSH}(\gamma)$$

for any $p > 0$. Since T_t is an L^q contraction for any q (Proposition 3.1), by the semigroup property the above implies that T_t is a contraction from L^p to L^q for any $q \geq e^{2t}p$. In other words, T_t is a contraction from L^p to L^q provided that $t \geq \frac{1}{2}\log(q/p)$, the Janson time $t_J(p,q)$. This is the strong hypercontractivity theorem proved in [15] for holomorphic functions on $\mathbb{C}^n \cong \mathbb{R}^{2n}$; here we prove it for LSH functions on $\mathbb{R}^n$.

Proof. The case where $f = \log|g|$ with g holomorphic on $\mathbb{C}^n$ is implicitly proved in [15] but is not given in this form. Using the ideas of Janson, we will prove the general theorem. Nelson's classical hypercontractivity result plays a crucial role here as in Janson's paper. Let $P_t = e^{-tN}$ be the Ornstein–Uhlenbeck semigroup. Let us write it in the form

$$P_t f(x) = \int M_r(x,y) f(y)\,\gamma(dy) \tag{3.2}$$

where $r = e^{-t}$ and M_r is the *Mehler kernel*

$$M_r(x,y) = (1-r^2)^{-n/2}\,\exp\left(-\frac{r^2}{1-r^2}|x|^2 + \frac{2r}{1-r^2}\langle x,y\rangle - \frac{1+r^2}{1-r^2}|y|^2\right). \tag{3.3}$$

We can rewrite Equation 3.2 in terms of Lebesgue measure as $P_t f(x) = \int K_r(x,y) f(y)\,dy$ where the modified kernel K_r is given by

$$K_r(x,y) = (1-r^2)^{-n/2}\,\exp\left(-\frac{|y-rx|^2}{1-r^2}\right).$$

Evidently $K_r(x,y)$ is constant in y on spheres around rx. This implies that if $f \geq 0$ is subharmonic, then for all $t > 0$ we have $P_t f(x) \geq f(e^{-t}x)$ (indeed, this is at the core of Janson's proof in [15]). The classical hypercontractivity inequality of Nelson (cf. [18]) is given by:

$$\|P_t f\|_{q(t),\gamma} \leq \|f\|_{p,\gamma}$$

where $q(t) = (p-1)e^{2t} + 1$ and $p > 1$. Hence, for $f \geq 0$ subharmonic, we have Nelson's theorem for the dilation semigroup:

$$\|f(e^{-t}x)\|_{q(t),\gamma} \leq \|f\|_{p,\gamma}. \tag{3.4}$$

Now take f to be LSH. The function $f^{1/p}$ is also LSH, so it is positive and subharmonic. Equation 3.4 applied to $f^{1/p}$ becomes

$$\left(\int f_{e^{-t}}(x)^{q(t)/p} d\gamma(x)\right)^{1/q(t)} \leq \left(\int f(x) d\gamma(x))\right)^{1/p}.$$

This implies that

$$\|f_{e^{-t}}\|_{q(t)/p,\gamma} \leq \|f\|_{1,\gamma}.$$

Observe that $\lim_{p\to\infty} \frac{q(t)}{p} = e^{2t} = \frac{1}{r^2}$ where $r = e^{-t}$. Applying Fatou's lemma, we obtain $\|f_r\|_{r^{-2},\gamma} \leq \|f\|_{1,\gamma}$, the desired result. □

In the full hypercontractivity theory due to Nelson [18], $t_N(p,q) = \frac{1}{2}\log\frac{q-1}{p-1}$ is the smallest time to contraction, for all L^p-functions. The analogous statement holds for Theorem 3.2; the exponent $1/r^2$ is optimal in this inequality (with Gaussian measure) over all LSH functions. In fact, it is optimal when restricted just to holomorphic functions on $\mathbb{C}^n$, as is proved (in an analogous non-commutative setting) in [17]; here we present a slightly different proof.

Proposition 3.4. *Let $r \in (0,1]$ and $C > 0$. Assume that for some $p > 0$, the following inequality holds for every* LSH *function f:*

$$\|f_r\|_{p,\gamma} \leq C\|f\|_{1,\gamma}. \tag{3.5}$$

Then $p \leq 1/r^2$ and $C \geq 1$.

Remark 3.5. If m is a probability measure then the L^p norm $\|f\|_{p,m}$ is a non-decreasing function of p. It follows that if Equation (3.5) holds for a $p > 1$ then it also holds for every $q \in [1,p)$.

Proof. Consider the set of functions $f^a(x) = e^{ax_1}$, which are all LSH for $a > 0$. An easy computation shows that $\|(f^a)_r\|_{p,\gamma} = \exp(r^2a^2p/2)$; in particular, $\|(f^a)\|_{1,\gamma} = \exp(a^2/2)$. The supposed inequality (3.5) then implies that $\exp(r^2a^2p/2) \leq C\exp(a^2/2)$ for all $a > 0$. Set $s = r^2p$. Then $\exp(a^2(s-1)/2) \leq C$ for every real a. Letting $a \to 0$ shows that $C \geq 1$; letting $a \to \infty$ shows that $s \leq 1$. □

Remark 3.6. Hypercontractive inequalities very typically involve actual contractions (i.e. constant $C = 1$ in Proposition 3.4), since the time constant (t_N or t_J in this case) are usually independent of dimension, yielding an infinite-dimensional version of the inequality. Indeed, in Nelson's original work [18], one main technique was to show that hypercontractivity held in all dimensions up to a fixed (dimension-independent) constant $C > 1$. The infinite-dimensional version then implies that $C = 1$ is the best inequality, for if the best constant is > 1 or < 1, a tensor argument shows that in infinite dimensions the constant is ∞ or 0, respectively.

We saw that the exponent $1/r^2$ is maximal in the (SHC) inequality for Gaussian measures. Below we show that it cannot be bigger for any probability measure with an exponential moment. In the following, $|x|$ refers to the Euclidean norm on $\mathbb{R}^n$.

Proposition 3.7. *Let μ be a probability measure with a finite exponential moment (i.e. $e^{c|x|}$ is μ-integrable for some $c > 0$) and such that for a linear form h on $\mathbb{R}^n$*

$$\int h(x)d\mu(x) = 0 \text{ and } \int h(x)^2 d\mu(x) \neq 0. \tag{3.6}$$

Fix $r \in (0,1)$. Assume that there exists $q(r) > 0$ such that

$$||f_r||_{q(r),\mu} \leq ||f||_{1,\mu} \tag{3.7}$$

for every LSH *function f. Then $q(r) \leq r^{-2}$.*

Remark 3.8. Observe that an $O(n)$-invariant probability measure with an exponential moment and not equal to δ_0 satisfies the condition (3.6).

Proof. One can assume that the μ-integral of h^2 is 1. Take the LSH function $f(x) = e^{\epsilon h(x)}$ where $\epsilon > 0$. The inequality (3.7) implies that

$$\int e^{\epsilon r q(r) h(x)} d\mu(x) \leq \left(\int e^{\epsilon h(x)} d\mu(x)\right)^{q(r)}. \tag{3.8}$$

Note that the last integral is finite for ϵ small enough, because μ has an exponential moment. Put $a = rq(r)$. We use the Taylor expansion $e^x = 1 + x + x^2/2 + g(x)$ where g satisfies: $|g(x)| \leq (|x|^3/6)e^{|x|}$. We get

$$\int e^{\epsilon a h(x)} d\mu(x) = 1 + \frac{a^2\epsilon^2}{2} + \int g(a\epsilon x) d\mu(x).$$

where the last term is $o(\epsilon^2)$. Similarly, we see that the right-hand side term of (3.8) can be written as $1 + q(r)\epsilon^2/2 + o(\epsilon^2)$. It follows that $a^2 \leq q(r)$, which means that $q(r) \leq r^{-2}$. □

4. HYPERCONTRACTIVITY INEQUALITIES FOR PROBABILITY MEASURES

In this section we study hypercontractivity properties of LSH functions with respect to any probability measure m. We have already seen in Proposition 3.1 that, for rotationally invariant measures m, the semigroup S_r is always an L^p contraction.

Theorem 4.1. *Fix $q > 1$ and $r \in (0, 1]$. Suppose that μ_1 and μ_2 are two probability measures on $\mathbb{R}^n$ which verify the hypercontractivity inequality*

$$\|f_r\|_{q,\mu} \leq \|f\|_{1,\mu} \tag{4.1}$$

for any continuous LSH *function f. It at least one of μ_1 and μ_2 is compactly-supported, then the convolved measure $\mu_1 * \mu_2$ also satisfies (4.1).*

Proof. Let f be a continuous LSH function, and suppose μ_1 is compactly-supported. We have

$$\begin{aligned} \int f(rz)^q d(\mu_1 * \mu_2)(z) &= \int \int f(rx + ry)^q \, d\mu_1(x) \, d\mu_2(y) \\ &\leq \int \left(\int f(x + ry) \, d\mu_1(x) \right)^q d\mu_2(y) \end{aligned}$$

since the function $x \mapsto f(x + ry)$ is continuous LSH for each fixed $y \in \mathbb{R}^n$, and μ_1 satisfies (4.1). Let $h(y) = \int f(x + y) \, d\mu_1(x)$, so that we have proven that

$$\|f_r\|^q_{q,\mu_1 * \mu_2} \leq \int h(ry)^q \, d\mu_2(y) = \|h_r\|^q_{1,\mu_2}. \tag{4.2}$$

Since f is continuous, the function $(x, y) \mapsto f(x + y)$ is continuous in both variables, and also LSH in each. Since $\operatorname{supp} \mu_1$ is compact and f is continuous, all the conditions of Corollary 2.4 are satisfied, and so h is LSH. Thence, by the assumption of the theorem, the quantity on the right-hand-side of Equation 4.2 is bounded above by $\|h\|^q_{1,\mu_2}$. By definition,

$$\|h\|_{1,\mu_2} = \int h(y) \, d\mu_2(y) = \int \int f(x + y) \, d\mu_1(x) \, d\mu_2(y) = \|f\|_{1,\mu_1 * \mu_2},$$

and this proves that Inequality 4.1 also holds for $\mu_1 * \mu_2$. □

The Theorem 4.1 suggests the following

Conjecture. *The convolution property of Theorem 4.1 holds without any assumptions on the measures μ_1, μ_2.*

It does not however seem easy to prove. This is due to the difficulty of proving that $f * \mu$ is upper semi-continuous when f is LSH, without any supplementary conditions on f or μ.

In the sequel we will only use Theorem 4.1 as stated, with μ_1 equal to a symmetric Bernoulli measure.

Most of the following results of this section concern the 1–dimensional case, i.e. log–convex functions on the real line. In that case, one has the following surprisingly general hypercontractivity inequality.

Proposition 4.2. *For every symmetric probability measure m on $\mathbb{R}$, and for any logarithmically convex function f on $\mathbb{R}$, the following inequality is true for any $r \in (0, 1]$:*

$$\|f_r\|_{1/r,m} \leq \|f\|_{1,m}.$$

Remark 4.3. Translating this statement into additive language, the dilation semigroup T_t satisfies strong hypercontractivity with time to contraction at most $2 \cdot t_J$, for any symmetric probability measure on $\mathbb{R}$, for log–convex functions. As explained above, a simple scaling $f \mapsto f^p$ yields the comparable result from $L^p \to L^q$ for $q \geq p > 0$.

Proof. By the log–convexity of f, for any $x \in \mathbb{R}$

$$f(rx) \leq f(0)^{1-r} f(x)^r,$$

which implies that $f(rx)^{1/r} \leq f(0)^{1/r-1} f(x)$. Then by m-integration,

$$\int f(rx)^{1/r} dm(x) \leq f(0)^{1/r-1} ||f||_{1,m}.$$

Since f is convex, $f(0) \leq \frac{1}{2}[f(x) + f(-x)]$ for all x. Integrating and using the symmetry of m yields $f(0) \leq ||f||_{1,m}$. Consequently,

$$\int f(rx)^{1/r} dm(x) \leq ||f||_{1,m}^{1/r},$$

and the Proposition follows. □

Remark 4.4. Proposition 4.2 remains true for rotationally invariant measures m and log–convex functions f on $\mathbb{R}^n$. This proof fails, however, for general LSH functions on $\mathbb{R}^n$ when $n \geq 2$.

Remark 4.5. Subject to additional regularity on m, the symmetry condition in Proposition 4.2 can be replaced with the much weaker assumption that m is centred: i.e. m has a finite first moment, and $\int x\, m(dx) = 0$. In short, fix a log-convex f, and suppose that m is regular enough that the function $\eta(r) = \int f(rx)\, m(dx)$ is differentiable, so that $\eta'(r) = \int f'(rx) x\, m(dx)$. (It is easy to see, from convexity of f, that $f_r \in L^1(m)$ for each r, provided $f \in L^1(m)$.) Then $\eta'(0) = f'(0) \int x\, m(dx) = 0$, and since f is convex, f' is increasing which means that $xf'(rx) \geq xf'(x)$ for all x, $r \geq 0$, so $\eta'(r) \geq \eta'(0) = 0$. Thus, $\int f\, dm = \eta(1) \geq \eta(0) = f(0)$, and the rest of the above proof follows. For this to work, it is necessary to assume (at minimum) that the functions $\frac{\partial}{\partial r} f(rx) = f'(rx)x$ are uniformly bounded in $L^1(m)$; a convenient way to achieve this is to assume that functions $g \in L^1(m)$ for which $x \mapsto xg'(x)$ is also in $L^1(m)$ are dense in $L^1(m)$. The kinds of measures for which such a Sobolev-space density is known is a main topic of our subsequent paper [8].

The problem in general is to find, for a fixed measure m, the maximal exponent q such that $\|f_r\|_{q,m} \leq \|f\|_{1,m}$ for every $r \in (0, 1]$ and any log-convex function f on $\mathbb{R}$. For symmetric Bernoulli measures we will show that the optimal exponent q is the same as for Gaussian measures.

Proposition 4.6. *If* $m = \frac{1}{2}(\delta_1 + \delta_{-1})$ *then*

$$\|f_r\|_{1/r^2,m} \leq \|f\|_{1,m} \tag{4.3}$$

for every $r \in (0, 1]$ *and any log-convex function* f.

Remark 4.7. It follows from Proposition 4.6, and a simple rescaling argument, that the same strong hypercontractivity inequality holds for any symmetric Bernoulli measure $\frac{1}{2}(\delta_a + \delta_{-a})$, $a > 0$. The optimality of the index $1/r^2$ in the inequality (4.3) follows from Proposition 3.7.

Proof. Step 1. We justify that it is sufficient to prove the proposition for the two-parameter family of functions $h(x) = C \exp(ax)$ with $a \in \mathbb{R}$ and $C > 0$. Take f strictly positive. Then there exists h of the form $C \exp(ax)$ such that the functions f and h are equal on the set $\{-1, +1\}$. Assume now that f is log-convex. Then $f \leq h$ on $[-1, 1]$, and in particular $f(r) \leq h(r)$ and $f(-r) \leq h(-r)$. This implies that

$$\int f(rx)^{1/r^2} dm(x) \leq \int h(rx)^{1/r^2} dm(x).$$

If the function h satisfies (4.3), we obtain

$$\|f_r\|_{q,m} \leq \|h_r\|_{q,m} \leq \|h\|_{1,m} = \|f\|_{1,m},$$

the last equality following from the fact that f and h coincide on the support of m. This gives the inequality (4.3) for f.

Step 2: We show the inequality (4.3) for $f(x) = e^{ax}$ (the constant C obviously factors out of the desired inequality). This is essentially an exercise. One has to prove that

$$\left(\int \exp(ax/r)dm(x)\right)^{r^2} \leq \int \exp(ax)dm(x),$$

i.e. $\left(\cosh(\frac{a}{r})\right)^{r^2} \leq \cosh a$ for a real and $r \in (0,1]$. Put $s = 1/r$. Then $s \geq 1$ and the required inequality becomes $\cosh(sa) \leq (\cosh a)^{s^2}$. Taking logarithms and next dividing by s^2a^2, we are left to prove that

$$\frac{\log(\cosh(sa))}{s^2a^2} \leq \frac{\log(\cosh a)}{a^2}.$$

In other words, we must prove that the function $\log(\cosh x)/x^2$ is decreasing for $x \geq 0$. Taking the derivative, it is sufficient to see that $\rho(x) = x \tanh x - 2\log(\cosh x)$ is nonpositive for $x \geq 0$. Well, $\rho(0) = 0$, and $\rho'(x) = x/\cosh^2 x - \tanh x = \frac{x - \sinh x \cosh x}{\cosh^2 x}$. This last quotient is non-positive for its numerator is equal to $x - (\sinh 2x)/2$. □

Remark 4.8. Proposition 4.3 could be obtained from an inequality of A. Bonami [3] similarly to the manner in which Theorem 3.2 was obtained from Nelson's hypercontractivity theorem for Gaussian measures. She proved that for symmetric Bernoulli measures the same classical hypercontractivity inequalities as for the Gaussian measure hold. In order to prove Proposition 4.3 for a log-convex function f, one compares it to the affine function which takes the same value as f on $\{-1,1\}$. For a function on $\{-1,1\}$, there is a unique affine function on the line which extends it. Thus one can identify the space $C\{-1,1\}$ of functions on $\{-1,1\}$ and the space of affine functions on the line. We omit the details.

Corollary 4.9. *The symmetric uniform probability measure λ_a on $[-a,a]$, $a > 0$, satisfies the strong hypercontractivity property $\|f_r\|_{1/r^2,\lambda_a} \leq \|f\|_{1,\lambda_a}$ for all* LSH *functions.*

Proof. Let $m_x = \frac{1}{2}(\delta_x + \delta_{-x})$. It is easy to see that

$$\mu_k := m_{\frac{1}{2}} * m_{\frac{1}{4}} * \ldots * m_{\frac{1}{2^k}} \rightharpoonup \lambda_1, \quad k \to \infty,$$

where we denote by $\rightharpoonup$ the convergence in law. By the Proposition 4.6 (and the proceeding Remark 4.7) and Theorem 4.1, the inequality (4.3) holds for the measures μ_k. The supports of the measures μ_k and λ_1 are compact and included in the segment $[-1,1]$. If f is log–convex on $\mathbb{R}$, it is continuous and the convergence $\int_{-1}^{1} f\, d\mu_k \to \int_{-1}^{1} f\, d\lambda_1$ follows from the convergence in law $\mu_k \Rightarrow \lambda_1$. The statement for all $a > 0$ now follows from a simple rescaling argument. □

Definition 4.1. For convenience, we introduce the notation (SHCc) for the *strong hypercontrativity coefficient* of a probability measure on $\mathbb{R}^n$: it is the supremum of the positive real numbers c such that for every LSH function f, one has the strong hypercontractivity inequality:

$$||f_r||_{r^{-c},\mu} \leq ||f||_{1,\mu} \quad r \in (0,1].$$

We have seen in Proposition 4.2 that for radial measures in dimension 1 the SHCc is at least 1. The following proposition shows that this is not true in higher dimensions.

Proposition 4.10. *In dimension bigger than one, the SHCc for radial measures can be* 0.

Proof. For clarity of notation here and in the following, let $N(x) = |x|$ denote the Euclidean norm on $\mathbb{R}^n$. Computing directly, one checks that for $n > 1$ this function is LSH (the laplacian of $\ln N(x)$ for x nonzero and observe also that $\ln N(0) = -\infty$). Then take a probability measure μ with a density $s(x) = 0$ for $N(x) \leq 1$ and of the form: $dN(x)^{-(n+2)}dx$ for $N(x) > 1$. The function $N(x)$ is LSH and integrable for this measure. But it is clear that for every positive value of c, the function $N(rx)^{r^{-c}}$ is not μ-integrable for r near 0. □

At the end of this Section we study the SHC properties for the probability measures

$$m_p(dx) = c_p \exp(-N(x)^p)dx, \quad p > 0.$$

By Theorem 3.2 and Proposition 3.4 we already know that for $p = 2$, the SHCc is 2 in any $\mathbb{R}^n$. Proposition 3.7 implies that for any $p > 0$, the SHCc is not greater than 2.

Proposition 4.11. **(a)** *In any dimension and for $p \geq 1$ the SHCc of the probability measure m_p is at most p.*

(b) *In dimension bigger than one, the result of Proposition 4.10 is true for any $p > 0$.*

(c) *For $p = 1$ and in dimension one, the SHCc is 1.*

Proof. **(a)** Take a function of the form $f(x) := \exp(AN(x)^p)dx$ with $0 < A < 1$. As $N(x)$ is convex and positive, the function $N(x)^p$ is also convex for $p \geq 1$ and then also SH. This implies that $\exp(AN(x)^p)$ is LSH. Moreover, it is m_p integrable. Fix r between 0 and 1. The integrability of the function $f(rx)^{q(r)}$ implies that $q(r) \leq r^{-p}$, which implies that the SHCc is at most p.

(b) The proof is the same as for (a), but one uses the fact that in dimension bigger than one, the function $N(x)^p$ is SH for every positive value of p.

(c) For the case $p = 1, n = 1$, one uses part **(a)** and the fact that in dimension one, the SHCc is at least one. □

Open question. Is the SHCc$= p$ for m_p when $1 < p < 2$ or when $p = 1$ and the dimension is bigger than 1?

5. LOGARITHMIC SOBOLEV INEQUALITIES FOR LSH FUNCTIONS

Recall that the classical Gaussian Logarithmic Sobolev Inequality, cf. [2, 9], is

$$\mathbf{Ent}(f^2) = \int |f|^2 \log |f|^2 d\gamma - \|f\|_{2,\gamma}^2 \log \|f\|_{2,\gamma}^2 \leq 2 \int fLf d\gamma = 2\mathcal{E}_L(f) \tag{5.1}$$

where γ is the standard Gaussian measure, $L = -\Delta + E$ is the generator of the Ornstein–Uhlenbeck semigroup and $f \in \mathcal{A}$, a standard algebra contained in the domain of the operator L. For the Ornstein–Uhlenbeck semigroup $\mathcal{A}$ can be chosen as the space of $\mathcal{C}^\infty$ functions with slowly increasing derivatives. The expression $\mathbf{Ent}(f)$ is called the *entropy* of f and $\mathcal{E}_L(f)$ is the *Dirichlet form* or *energy* of f, with respect to the generator L of the Ornstein-Uhlenbeck semigroup, cf.[2].

The celebrated theorem of Gross [9] establishes the equivalence between the hypercontractivity property of a semigroup T_t with invariant measure μ and the log–Sobolev inequality relative to the generator L of T_t. More precisely, recalling the Nelson time $t_N = \frac{1}{2} \ln \frac{q-1}{p-1}$, the hypercontractivity inequalities $\|T_t f\|_{q,\mu} \leq \|f\|_{p,\mu}$ for $t \geq c\, t_N(p,q)$ for $1 < p \leq q < \infty$ are, together, equivalent to the single log–Sobolev Inequality

$$\mathbf{Ent}(f^2) = \int |f|^2 \log |f|^2 d\mu - \|f\|_{2,\mu}^2 \log \|f\|_{2,\mu}^2 \leq 2c \int fLf\, d\mu = 2c\mathcal{E}_L(f). \tag{5.2}$$

In the Gaussian case these inequalities indeed hold with $c = 1$.

In this section we will prove that a **strong** Log-Sobolev Inequality

$$\mathbf{Ent}(f^2) = \int |f|^2 \log |f|^2 d\mu - \|f\|_{2,\mu}^2 \log \|f\|_{2,\mu}^2 \leq c \int fEf\, d\mu = c\mathcal{E}_E(f) \tag{5.3}$$

holds for log–subharmonic functions f and compactly supported measures μ for which a (SHC) property holds. As the Dirichlet form (or energy) on the right-hand side of (5.3) are taken with respect to the generator E of the considered dilation semigroup $T_t f(x) = f(e^{-t}x)$, the inequality (5.3) may also be called an *Euler type* LSI.

Observe that all the above-mentionned inequalities have L^1 versions. If in (5.1) we consider $f > 0$ and we substitute $f = \sqrt{g}$, then using the formulas $\int fLf d\gamma = \int (\nabla f)^2 d\gamma$ and $\nabla f = \frac{\nabla g}{2\sqrt{g}}$ we get

$$\mathbf{Ent}(g) \leq \frac{1}{2}\int \frac{(\nabla g)^2}{g} d\gamma. \tag{5.4}$$

Let f be LSH, and set $g = f^2$ in (5.3). Using the fact that $Eg = 2fEf$ we can write the inequality (5.3) as

$$\mathbf{Ent}(g) = \int g \log g \, d\mu - \|g\|_{1,\mu} \log \|g\|_{1,\mu} \leq \frac{c}{2}\int Eg \, d\mu. \tag{5.5}$$

It may seem surprising that the integrals $\int fEf \, d\mu$ from (5.3) and, equivalently, $\int Eg \, d\mu$ from (5.5) are positive when f and g are LSH functions. The following proposition explains this phenomenon, which holds more generally for subharmonic functions.

Proposition 5.1. *Let m be a probability measure on $\mathbb{R}^n$ which is $O(n)$ invariant, and let $g \in \mathcal{C}^1$ be a subharmonic function. Then*

$$I = \int Eg(x) dm(x) \geq 0.$$

Proof. We have

$$I = \int dm(x) \int_{O(n)} Eg(\alpha x) \, d\alpha,$$

where $d\alpha$ denotes the Haar measure on $O(n)$. Denote by σ the normalized Lebegue measure on the unit sphere S^{n-1}. If $r = \|x\|$, we have

$$\begin{aligned}\int_{O(n)} Eg(\alpha x) \, d\alpha = \int_{S^{n-1}} (Eg)(ru) \, \sigma(du) &= r\int_{S^{n-1}} \frac{\partial g}{\partial r}(ru) \, \sigma(du) \\ &= r\frac{\partial}{\partial r}\int_{S^{n-1}} g(ru) \, \sigma(du) \geq 0\end{aligned}$$

because the function $r \mapsto \int_{S^{n-1}} g(ru) \, \sigma(du)$ is increasing (cf. Corollary 2.7). □

5.1. **Log-Sobolev Inequalities for measures with compact support.** The following techniques work, in principle, quite generally. However, the usual approximation techniques to guarantee integrability (convolution approximations and cut-offs) are unavailable in the category of subharmonic functions. As such, we include this section which develops the relevant log-Sobolev inequalities in all dimensions, but only for compactly–supported measures (i.e. do the cut-off in the measure rather than the test functions). Extension of these results to a much larger class of measures is the topic of [8].

Theorem 5.2. *Let μ be a probability measure on $\mathbb{R}^n$ with* **compact support**. *Suppose that for some $c > 0$, the following strong hypercontractivity property holds: for $0 < p \leq q < \infty$ and $f \in L^p_{\mathrm{LSH}}(\mu)$,*

$$\|f_{e^{-t}}\|_{q,\mu} \leq \|f\|_{p,\mu} \quad \text{for } t \geq c \cdot \tfrac{1}{2}\log\tfrac{q}{p}.$$

Then for any log–subharmonic function $f \in \mathcal{C}^1$ the following logarithmic Sobolev inequality holds:

$$\int f^2 \log f^2 d\mu - \|f\|^2_{2,\mu} \log \|f\|^2_{2,\mu} \leq c\int fEf d\mu. \tag{5.6}$$

Remark 5.3. (1) The condition $f \in \mathcal{C}^1$ is natural to ensure a good sense of the expression Ef in (5.6). In the classical case in [2] one supposes $f \in \mathcal{A} \subset \mathcal{C}^\infty$ and such an LSI inequality is equivalent to the hypercontractivity property ([2], Theorem 2.8.2).

(2) In the case of strong hypercontractivity with optimal $q = p/r^2$ (symmetric Bernoulli measures and their convolutions, symmetric uniform measures on $[-a, a]$), the constant c is equal to 1. Also Gaussian measures on $\mathbb{R}^n$ have the constant $c = 1$ but evidently they are not covered by the Theorem 5.2. When $q = p/r$ (any symmetric measure on $\mathbb{R}$), the constant c is equal to 2. The time $t_J = \frac{1}{2}\log\frac{q}{p}$ appearing in Theorem 5.2 is Janson's time.

(3) Theorem 5.2 is stated and proved here for compactly-supported measures, a class not including the most important Gaussian measures. In the end of this section we will show that the same strong log-Sobolev inequality of Euler type holds for Gaussian measures in all dimensions.

Let us reiterate that the following proof applies to a much wider class of measures, but the precise regularity conditions are complicated by the fact that cut-off approximations do not preserve the cone of log–subharmonic functions. This will be covered in [8].

Proof. Let $p = 2$ and t be the critical time $t = c \cdot \frac{1}{2}\log\frac{q}{p}$. Then the variable $r = e^{-t}$ satisfies $q(r) = 2r^{-2/c}$. The method of proof is classical and consists of differentiating the function

$$\alpha(r) = \|f_r\|_{q(r),\mu}$$

at $r = 1$. By strong hypercontractivity, $\alpha(r) \le \alpha(1)$, so $\alpha'(1) \ge 0$ if we prove the existence of this derivative.

Define $\beta(r) = \alpha(r)^{q(r)} = \int f(rx)^{q(r)} d\mu(x)$ and let $\beta_x(r) = f(rx)^{q(r)}$, so that $\beta(r) = \int \beta_x(r) d\mu(x)$. Then

$$\frac{\partial}{\partial r}\log\beta_x(r) = q'(r)\log f(rx) + \frac{q(r)}{f(rx)} x \cdot \nabla f(rx).$$

Since $q'(r) = -\frac{2}{rc}q(r)$, we compute

$$\beta_x'(r) = -\frac{2}{rc} f_r(x)^{q(r)} \log f_r(x)^{q(r)} + \frac{q(r)}{r} f_r(x)^{q(r)-1}(Ef)_r(x). \tag{5.7}$$

Let $0 < \epsilon < 1$. As $f \in \mathcal{C}^1$, the expression on the right-hand side of (5.7) is bounded for $r \in (1-\epsilon, 1]$ and $x \in \operatorname{supp}(\mu)$ (which is compact). The Dominated Convergence Theorem then implies that

$$\beta'(r) = \frac{\partial}{\partial r}\int \beta_x(r)\, d\mu(x) = \int \beta_x'(r) d\mu(x). \tag{5.8}$$

Finally, since $\alpha(r) = \beta(r)^{1/q(r)}$ and $\beta > 0$, we have that α is $\mathcal{C}^1$ on $(1-\epsilon, 1]$ and a simple calculation shows that

$$\alpha'(r) = \frac{\alpha(r)}{q(r)\beta(r)}\left[\frac{2}{rc}\beta(r)\log\beta(r) + \beta'(r)\right].$$

Now, taking $r = 1$, applying $\alpha'(1) \ge 0$ and the formulas (5.7) and (5.8) we obtain

$$\begin{aligned} 0 &\le \frac{2}{c}\beta(1)\log\beta(1) + \beta'(1) \\ &= \frac{2}{c}\|f\|_{2,\mu}^2 \log\|f\|_{2,\mu}^2 - \frac{2}{c}\int f^2 \log f^2 d\mu + 2\int fEf d\mu, \end{aligned}$$

and this is the logarithmic Sobolev inequality (5.6). ☐

For $p > 0$ we define spaces $L_E^p(\mu) = \{f\,;\, f \in L^p(\mu) \text{ and } Ef \in L^p(\mu)\}$ and $L^p(\mu)\log L^p(\mu) = \{f\,;\, \int f^p|\log f^p| d\mu < \infty\}$ (we think that the notation $L^p \log L^p$ is more appropriated than $L^p \log L$ that can sometimes be met). The former is a Sobolev space, the latter an Orlicz space, related to the logarithmic Sobolev inequality 5.6; indeed, in the case $p = 2$, they are the spaces for which the right– and left–hand sides (respectively) of that inequality are finite.

Appealing to the surprising Proposition 4.2, and Theorem 5.2, we have the following.

Corollary 5.4. *Let μ be a symmetric probability measure on $\mathbb{R}$. Then for any log-subharmonic function $f \in L^2(\mu)\log L^2(\mu) \cap L_E^2(\mu) \cap \mathcal{C}^1$ the following logarithmic Sobolev inequality holds:*

$$\int f^2 \log f^2 d\mu - \|f\|_{L^2(\mu)}^2 \log\|f\|_{L^2(\mu)}^2 \le 2\int fEf d\mu.$$

Remark 5.5. In the classical case it is sufficient to suppose only $f \in L_E^2(\mu)$; this actually implies that $f \in L^2(\mu)\log L^2(\mu)$. The proof of this fact involves approximation by more regular (e.g. compactly supported or bounded) functions, and these tools are unavailable to us here.

Proof. By Proposition 4.2 the measure μ as well as the measures $\mu_N = \mu|_{[-N,N]} + \mu([-N,N]^c)\delta_0$ verify the strong hypercontractivity property for LSH functions with $q = p/r$ and $c = 2$. Let f verify the hypothesis of the corollary, and set $f^\epsilon = f + \epsilon$; it is easy to check that f^ϵ also verifies all the conditions of the corollary. By Theorem 5.2, for each N

$$\int (f^\epsilon)^2 \log(f^\epsilon)^2 d\mu_N - \|f^\epsilon\|^2_{2,\mu_N} \log \|f^\epsilon\|^2_{2,\mu_N} \leq 2 \int f^\epsilon E f^\epsilon d\mu_N.$$

When $N \to \infty$, $\mu_N \rightharpoonup \mu$ (weak convergence), and since $f^\epsilon \in \mathcal{C}^1$ and is strictly positive, all the functions $(f^\epsilon)^2$, $(f^\epsilon)^2 \log(f^\epsilon)^2$, and $f^\epsilon E f^\epsilon$ are continuous; hence the integrals in the last formula converge to analogous integrals in terms of f^ϵ with respect to the measure μ. Finally, we can let $\epsilon \downarrow 0$ to achieve the result, by the Monotone Convergence Theorem. □

Corollary 5.6. *Let μ be a symmetric probability measure on $\mathbb{R}$. Then for any log–subharmonic function $f \in L^1(\mu) \log L^1(\mu) \cap L^1_E(\mu) \cap \mathcal{C}^1$ the following logarithmic Sobolev inequality holds:*

$$\int f \log f d\mu - \|f\|_{1,\mu} \log \|f\|_{1,\mu} \leq \int E f d\mu.$$

Proof. The proof is similar to the proof of the Corollary 5.4. Note, nevertheless, that Corollary 5.6 does not follow from Corollary 5.4 because the hypothesis $Ef \in L^1(\mu)$ is weaker than the condition $Ef \in L^2(\mu)$ supposed in Corollary 5.4 (all other integrability hypotheses are equivalent by the transformation $f \mapsto f^2$ which maps L^2 onto L^1). □

5.2. **Log-Sobolev Inequality for Gaussian measures.** We formulate two versions of the strong Logarithmic Sobolev Inequality for log-subharmonic functions and Gaussian measures: in the classical context $L^2(\gamma)$ (Theorem 5.7) and in the more natural and technically simpler case $L^1(\gamma)$ (Theorem 5.8).

Both cases are nearly equivalent since $f \in L^2(\gamma)$ and log–subharmonic is equivalent to $f^2 \in L^1(\gamma)$ and log–subharmonic. But the integration hypotheses of the theorems are slightly different, cf. the discussion in the proof of the Corollary 5.6.

Theorem 5.7. *Let γ be the Gaussian measure with density $\frac{1}{\sqrt{(2\pi)^n}} e^{-|x|^2/2}$ on $\mathbb{R}^n$. Then for any* LSH *and $\mathcal{C}^1$ function $f \in L^2(\gamma) \log L^2(\gamma) \cap L^2_E(\gamma)$ the following logarithmic Sobolev inequality holds*

$$\int f^2 \log f^2 d\gamma - \|f\|^2_{2,\gamma} \log \|f\|^2_{2,\gamma} \leq \int f E f d\gamma. \tag{5.9}$$

Theorem 5.8. *Let γ be as in Theorem 5.7. Then for any* LSH *and $\mathcal{C}^1$ function $g \in L^1(\gamma) \log L^1(\gamma) \cap L^1_E(\gamma)$ the following logarithmic Sobolev inequality holds*

$$\int g \log g d\gamma - \|g\|_{1,\gamma} \log \|g\|_{1,\gamma} \leq \frac{1}{2} \int E g d\gamma. \tag{5.10}$$

Note that the method of the proof of Corollary 5.4 cannot be applied because we do not know if the measures γ_N have the strong hypercontractivity property with Gaussian constant $c = 1$; by the Theorem 4.2 they have it with $c = 2$ and we would obtain a weaker inequality with the constant 2 before the energy term $\mathcal{E}_E(f)$. Instead, we will use the classical LSI for Gaussian measures.

Proof. Let us prove Theorem 5.8; the proof of Theorem 5.7 is similar.

It is sufficient to consider the case $g = \exp(h)$ with $h \in \mathcal{C}^2$ and $\Delta h \geq 0$. It follows that

$$\Delta g \geq \frac{(\nabla g)^2}{g}$$

which combined with the L^1 version of the classical LSI (5.4) gives

$$\mathbf{Ent}(g) \leq \frac{1}{2} \int \Delta g d\gamma.$$

We also have

$$\int E g d\gamma = \int \Delta g d\gamma.$$

Finally

$$\mathbf{Ent}(g) \le \frac{1}{2}\int E g d\gamma$$

what is our strong LogSI(5.10). □

Remark 5.9. For the log–subharmonic functions $f(x) = e^{ax}, a > 0$ there is equality in (5.9) and (5.10). Thus the constant $c = 1$ is optimal in (5.9) and the constant $\frac{1}{2}$ is optimal in (5.10).

Remark 5.10. Let $m = \frac{1}{2}(\delta_1 + \delta_{-1})$ and let μ_k denote the normalized convolution powers m^{*k}. By the Central Limit Theorem(CLT), the measures μ_k converge in law to γ. As the Theorem 5.2 applies to the measures μ_k, one can prove the strong LSI's for the measure γ on $\mathbb{R}$ using a strengthened version of the CLT, cf. [7]. Strong LSI's are proven in [7] also for Gaussian measures γ on $\mathbb{R}^n$, $n \ge 2$, using approximation of γ by uniform spherical measures. This approach mirrors, to some extent, Gross's proof of the Gaussian log-Sobolev inequality in [9].

A direct proof of SHC inequality for γ using Proposition 4.6 is also a corollary of results of [7].

Remark 5.11. In principle, the strong LSI for Gaussian measures or other non-compactly supported measures should follow from the strong hypercontractivity inequalities of Theorem 3.2 via an approach like that in the proof of Theorem 5.2. As we have mentioned, there are challenging regularization issues (due to the nature of logarithmically subharmonic functions) which complicate these techniques. Along the same lines, any measure for which the Logarithmic Sobolev Inequality holds for LSH functions should also satisfy strong hypercontractive estimates (this was proved in the restricted context of holomorphic functions in [10]). Thus an equivalence

$$\text{SHC} \iff \text{strong LSI}$$

is a natural conjecture. These issues will be dealt with in a future publication [8].
Other important open problems to be studied are:
– proving SHC for semigroups with other generators L
– SHC inequalities for non-symmetric Bernoulli and uniform measures
– a general convolution property, weakening the strong assumptions of Theorem 4.1.

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(1) UNIVERSITÉ D'ANGERS, 2 BOULEVARD LAVOISIER, 49045 ANGERS CEDEX 01, FRANCE
E-mail address: Piotr.Graczyk@univ-angers.fr, Jean-Jacques.Loeb@univ-angers.fr

(2) 2-175, MIT 77 MASSACHUSETTS AVENUE, CAMBRDIGE, MA 02139
E-mail address: tkemp@math.mit.edu