Where do we go from here?

This course has been an introduction to modern probability theory. We have barely scratched the surface of Probability, and with few exceptions (like Stein's method), we've introduced theory that has been well-understood and standardized since the 1940s (or earlier).

It would be difficult to give a "comprehensive list" of probability topics out there today.

Instead, I wanted to leave you with thoughts on a few natural next steps for subjects / courses that follow naturally from where we've ended up.

* **Continuous-Time (Sub) Martingales**

$(X_t)_{t\geq 0} \in L^1$, $\mathbb{E}[X_t - X_s \mid \mathcal{F}_s] = 0$ $(\geq)$ $\forall s<t$.

↳ Convergence theorems ($\lim_{t\to\infty} X_t$, $\lim_{t\to t_0} X_t$ under L^1-bounded, unif. integrability)

↳ Optional Stopping, Optional Sampling

↳ Maximal / L^p inequalities

↳ Regularization; filtration augmentation and right continuity

↳ Applications to (more) properties of Brownian motion.

(This could have been done in this course, with 2-3 more weeks)

* **Stochastic Integration**

Martingale $(M_t)_{t\geq 0}$; progressive process $(X_t)_{t\geq 0}$

↳ $Y_t = Y_0 + \int_0^t X_s \, dM_s$

"$dY_t = X_t \, dM_t$."

$\cancel{\dfrac{dY_t}{dM_t} = X_t}$

More generally $dY_t = X_t \, dM_t + Z_t \, dt$

↳ Itô's formula: for $f \in C^2$, $d f(X_t) = f'(X_t)\,dX_t + \frac{1}{2} f''(X_t)\,d\underbrace{[X]_t}$

quadratic var.

$[B_t] = t$.

↳ Applications to stochastic processes

Eg. • If $(M_t)_{t\geq 0}$ is a continuous martingale,
then there is a Brownian motion B s.t. $M_t = B_{[M]_t}$

Cor: M-paths are locally Brownian i.e. C^α for $\alpha < \frac{1}{2}$, a.s. not loc. C^α for any $\alpha > \frac{1}{2}$.

• Feynman-Kac formula: if $V: \mathbb{R}^d \to \mathbb{R}$ is a "nice" potential,
the PDE

$$\begin{bmatrix} \partial_t u = \frac{1}{2}\Delta u - V \cdot u \\ u(0,x) = f(x) \end{bmatrix}$$

has unique solution

$$u(x,t) = E^x\left[f(B_t)\, e^{-\int_0^t V(B_s)\,ds} \right].$$

* Stochastic Differential Equations

$$dX_t = \sigma(t, X_t)\, dB_t + \mu(t, X_t)\, dt$$ ← Stohastic ODE

(Fairly well-defined, well understood existence / uniqueness)

Stochastic PDEs : much harder, even to make sense of.

↳ Connections to random surfaces,
random geometry,
statistical physics, ...

* Feller - Markov Processes

↳ Markov processes in locally compact metric space S

take a Funct-Analysis Corse:

s.t. 1. $Q_t(C_0(S)) \subseteq C_0(S)$

2. $\|Q_t f - f\|_\infty \to 0$ as $t \downarrow 0$ for $f \in C_0(S)$.

E.g. Brownian motion, Poisson process.

Continuous-time homogeneous Markov processes with generators $Q_t = e^{tA}$ A densely-defined in $C_0(S)$.

Then there are lots of topics that continue from points throughout this course.

* Renewal processes, birth and death processes
* Queueing systems
* Large Deviations
* Stein's Method
* Entropy
* Stable distributions
* Lévy processes

And there are probability and related fields that are very "hot" right now, including:

* Random Matrix Theory (and Free Probability)
* Random Graphs and Networks
* Random fields and surface growth (KPZ, SLE)

And ...