We've seen that, on $\mathbb{R}$, we can exactly calculate the (distribution of the) Hitting time of any height $r>0$:

$$\mathbb{P}^0(T_r \le t) = \mathbb{P}(|B_t| \ge r) = \int_0^t \frac{r}{\sqrt{2\pi u^3}} e^{-r^2/2u}\, du$$

What about in higher dimensions?

For $B.$ on $\mathbb{R}^d$, let T_R = Hitting time of $D_R(0)^c$ $= \inf\{t>0 : |B_t| > R\}$

disk $\{x \in \mathbb{R}^d : |x| \le R\}$

In [Lec. 55.2], we showed that

$$\limsup_{t\to\infty} \frac{|B_t|}{t^\alpha} = \infty \quad \forall \alpha \in (0, \tfrac{1}{2}) \quad \mathbb{P}^0\text{-a.s.}$$

$\therefore \mathbb{P}^0(T_R < \infty) = 1 \quad \forall R > 0$

Note: the $\mathbb{P}^x$-law of $B.$ is equal to the $\mathbb{P}^0$-law of $x + B.$

So we can always translate questions about balls $D_R(x)$ to $D_R(0)$.

Theorem: Let $D \subseteq \mathbb{R}^d$ be open, and let $\tau_D = \inf\{t \geq 0 : B_t \in D^c\}$. Debut time of D^c
↑ closed.

[Lec 56.2] stopping time.

Let $f : \partial D \to \mathbb{R}$ be bounded and measurable. Define

$$u : D \to \mathbb{R} \; ; \qquad u(x) = E^x[\, f(B_{\tau_D}) : \tau_D < \infty \,].$$

Then $u \in C^\infty(D)$, and $\Delta u = 0$.

on $\{\tau_D < \infty\}$. $B_{\tau_D} \in D^c$

But $B_t \in D \;\; \forall t < \tau_D$.

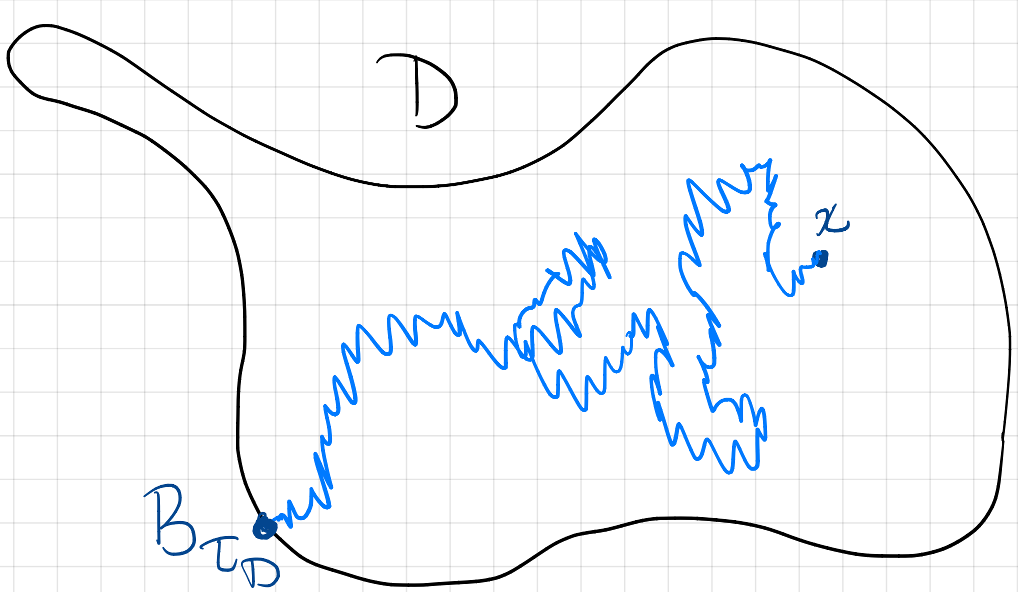
$\therefore \; B_{\tau_D} \in \partial D$.

I.e. u is a solution to the Dirichlet Problem

$$\begin{bmatrix} \Delta u = 0 & \text{on } D \\ u = f & \text{on } \partial D \end{bmatrix}$$

← usually want continuity.

(Use probability theory to show this PDE has a solution $\forall$ bounded measurable f, no matter how rough!)



← An example of a "path integral" expressing the solution of a PDE. For a closer analog to QFT, see Feynman-Kac.

For the proof, we make use of the following Real Analysis fact: (due to Gauss)

Theorem: If $D \subseteq \mathbb{R}^d$ is open, a measurable function $u: D \to \mathbb{R}$ is **harmonic** ($u \in C^\infty(D)$ and $\Delta u = 0$ on D) iff it has the **mean-value property**: $\forall x \in D$ and all $R > 0$ s.t. $\mathbb{D}_R(x) \subset D$,

$$u(x) = \int_{\partial \mathbb{D}_R(x)} u(y)\, \nu_{\partial \mathbb{D}_R(x)}(dy)$$

← uniform surface prob. measure on $\partial \mathbb{D}_R(x)$.
"Haar measure"

We use this in concert with the following observation:

If $U \in SO(d)$ rotation of $\mathbb{R}^d$
then $B^U_t := U \cdot B_t$ is a Brownian motion. } check covariance.

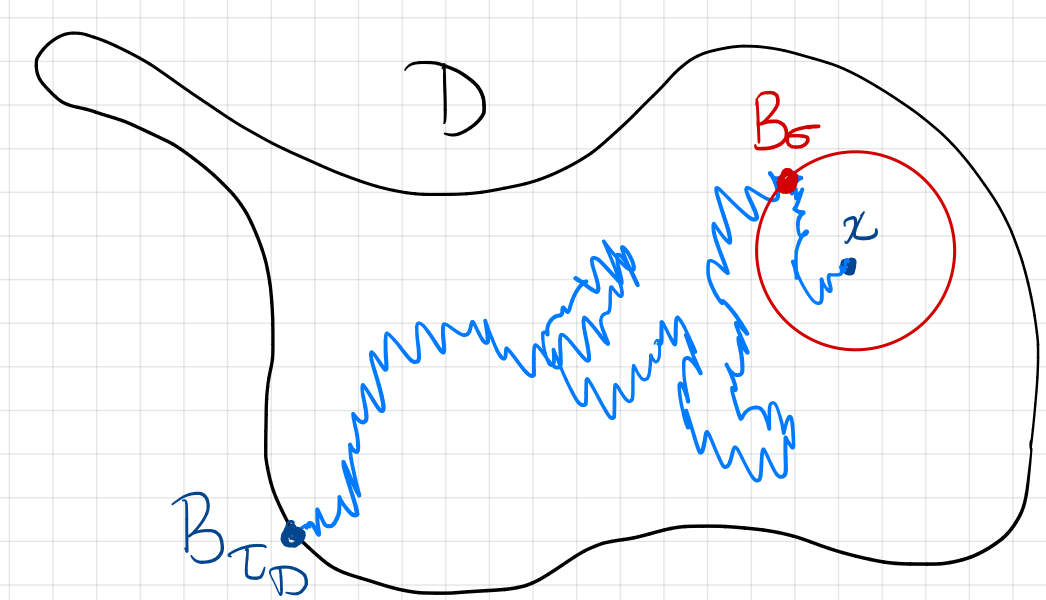
Cor: Under $\mathbb{P}^x$, $B_{\tau_{\mathbb{D}_R(x)}} \overset{d}{=} \nu_{\partial \mathbb{D}_R(x)}$.

Pf. We know $B_{\tau_{\mathbb{D}_R(x)}} \in \partial \mathbb{D}_R(x)$. For $E \in \mathcal{B}(\partial \mathbb{D}_R(x))$,

$$\mathbb{P}^x(B_{\tau_{\mathbb{D}_R(x)}} \in U^{-1} E) = \mathbb{P}^x\big((U \cdot B)_{\tau_{\mathbb{D}_R(x)}} \in E \big)$$
$$\overset{d}{=} B$$

$\therefore$ Law$(B_{\tau_{\mathbb{D}_R(x)}})$ is $SO(d)$-invariant. ///

Pf. (of connection to the Dirichlet problem)



Let $x \in D$. By definition

$$u(x) = \mathbb{E}^x[f(B_{\tau_D})\mathbb{1}_{\{\tau_D<\infty\}}] = \mathbb{E}^x[F(B_\cdot)]$$

$$F : C([0,\infty),\mathbb{R}^d) \to \mathbb{R}$$

$$F(\omega) = f(\omega(\tau_D^\omega))\mathbb{1}_{\{\tau_D^\omega<\infty\}}$$

If $x \in \partial D$, $\tau_D = 0$ $\mathbb{P}^x$-a.s. and

$\therefore u(x) = \mathbb{E}^x[f(B_0)] = f(x)$.

Now, D is open, so a nbhd of x is in D.

Let $R>0$ be any radius s.t. $D_R(x) \subset D$. We know $\tau_{D_R(x)} < \infty$ $\mathbb{P}^x$-a.s.

$=: \sigma$.

Any path from x to ∂D passes through $\partial D_R(x)$.

$$\therefore F(\omega\circ\theta_{\sigma^\omega}) = f(\underbrace{\omega\circ\theta_{\sigma^\omega}(\tau_D^{\omega\circ\theta_\sigma})}_{\text{the point on } \partial D \text{ is the same; just reached sooner.}})\mathbb{1}_{\{\tau_D^{\omega\circ\theta_\sigma^\omega}<\infty\}}$$

$$= F(\omega).$$

$$\therefore = \mathbb{E}^x[F(B)] = \mathbb{E}^x[F(B\circ\theta_\sigma)]$$

$u(x) = \mathbb{E}^x[F(B_\cdot)] = \mathbb{E}^x[F(B_{\sigma+\cdot})]$ where $F(\omega) = f(\omega(\tau_D))\mathbb{1}_{\{\tau_D<\infty\}}$

$= \mathbb{E}^x[\mathbb{E}^x[F(B_{\sigma+\cdot})|\mathcal{F}_\sigma]]$ $\quad \sigma = \tau_{D_R(x)}$

$= \mathbb{E}^x[\mathbb{E}^y[F(B_\cdot)]|_{y=B_\sigma}]$

$= \mathbb{E}^x[u(B_\sigma)]$ $\quad$ Law $(B_\sigma) = \nu_{\partial D_R(x)}$ under $\mathbb{P}^x$.

$= \int_{\partial D_R(x)} u \, d\nu_{\partial D_R(x)}.$

///

PDE tools can be used to show that if ∂D is sufficiently regular, u is in $C(\bar{D})$.

Indeed, the easier statement of the Dirichlet problem is: if $u, v \in C(\bar{D})$ and $\Delta u = \Delta v = 0$ on D, then $u = v$.

(Follows from the Maximum Principle.)

The upshot is: if we can <u>find</u> a function $\bar{u} \in C(\bar{D})$ s.t. $u = \bar{u}|_D$ is harmonic, then we must have

$$u(x) = \mathbb{E}^x[\bar{u}(B_{\tau_D})\mathbb{1}_{\{\tau_D<\infty\}}] \quad \forall x \in D.$$

For $d \geq 2$, define $u_d : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ by

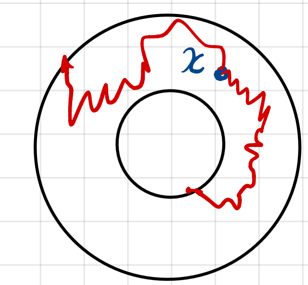
$$\hat{u}_d(|x|) = u_d(x) = \begin{cases} \log|x| & d=2 \\ |x|^{2-d} & d \geq 3 \end{cases}$$

Check that $\Delta u_d = 0$, $u_d \in C^\infty(\mathbb{R}^d \setminus \{0\})$ (calculus HW)

Now, let $0 < r < R < \infty$, and set $D_{r,R} = \{x \in \mathbb{R}^d : r < |x| < R\}$

Let T_r = Hitting time of $\partial D_r(0)$; then $\underbrace{\tau_{D_{r,R}}}_{=\tau} = \inf\{t \geq 0 : B_t \notin D_{r,R}\}$

$$= \min\{T_r, T_R\}$$



$$\therefore u_d(x) = E^x[u_d(B_\tau)] \quad \forall x \in D_{r,R}$$

$$= E^x[\underbrace{u_d(B_\tau)}_{\hat{u}_d(r)} : T_r < T_R] + E^x[\underbrace{u_d(B_\tau)}_{\hat{u}_d(R)} : T_r > T_R]$$

$$= \hat{u}_d(r)\, \mathbb{P}^x(T_r < T_R) + \hat{u}_d(R)\left(1 - \mathbb{P}^x(T_r < T_R)\right)$$

Rearranging this, we find:

$$\mathbb{P}^x(T_r < T_R) = \frac{\hat{u}_d(R) - \hat{u}_d(|x|)}{\hat{u}_d(R) - \hat{u}_d(r)} = \begin{cases} \dfrac{\log R - \log|x|}{\log R - \log r}, & d=2 \\[2ex] \dfrac{R^{2-d} - |x|^{2-d}}{R^{2-d} - r^{2-d}}, & d \geq 3 \end{cases}$$

Cor: In $\mathbb{R}$, Brownian motion is recurrent: $\mathbb{P}^x(T_0<\infty)=\mathbb{P}^0(T_x<\infty)=1\ \forall x\in\mathbb{R}$.

In $\mathbb{R}^2$, Brownian motion is neighborhood recurrent: $\mathbb{P}^x(T_r<\infty)=1\ \forall r>0$

(but $\mathbb{P}^x(T_0<\infty)=0$)

In $\mathbb{R}^d$ for $d\geq 3$, Brownian motion is neighborhood transient:

$\mathbb{P}^x(T_r<\infty)<1\ \forall r>0$.

Pf. $(d=1)$ Done before. ✓ (OR follow the $d=2$ case, $u_1(x)=x$)

$(d\geq 2)$ Note that $\limsup_{t\to\infty}\frac{|B_t|}{t^\alpha}=0$ $\mathbb{P}^x$-a.s. for $\alpha>\frac{1}{2}$.

$\therefore T_R=\inf\{t\geq 0: |B_t|=R\}\geq R^{1/\alpha}$ a.s. $\forall$ large R,

$\to\infty$ as $R\to\infty$.

$\therefore \forall r>0$, $\mathbb{P}^x(T_r<\infty)=\lim_{R\to\infty}\mathbb{P}^x(T_r<T_R)$

- $d=2$: $\quad =\lim_{R\to\infty}\frac{\log R-\log|x|}{\log R-\log r}=1$.

- $d\geq 3$: $\quad =\lim_{R\to\infty}\frac{R^{2-d}-|x|^{2-d}}{R^{2-d}-r^{2-d}}=\left(\frac{r}{|x|}\right)^{d-2}<1$

Also: $\mathbb{P}^x(T_0<T_R)=\lim_{r\downarrow 0}\mathbb{P}^x(T_r<T_R)=0$.

$\therefore T_0\geq T_R$ a.s. $\forall R>0$. $T_R\to\infty$ as $R\to\infty$. ///