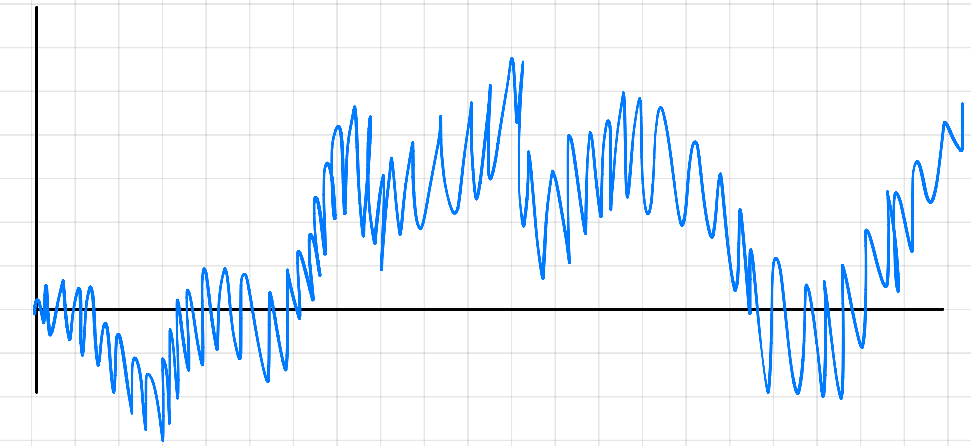
Let $(B_t)_{t\geq 0}$ be a Brownian motion on $\mathbb{R}$.
Consider the running maximum $B_t^{max} = \max_{s\leq t} B_s$.



For discrete-time submartingales, we proved several useful estimates for the values of running maxima

we could (given more time) prove the same results for continuous-time martingales like Brownian motion. But we can prove stronger results here.

Theorem: (Bachelier) For each $t\geq 0$, $B_t^{max} \overset{d}{=} |B_t|$.

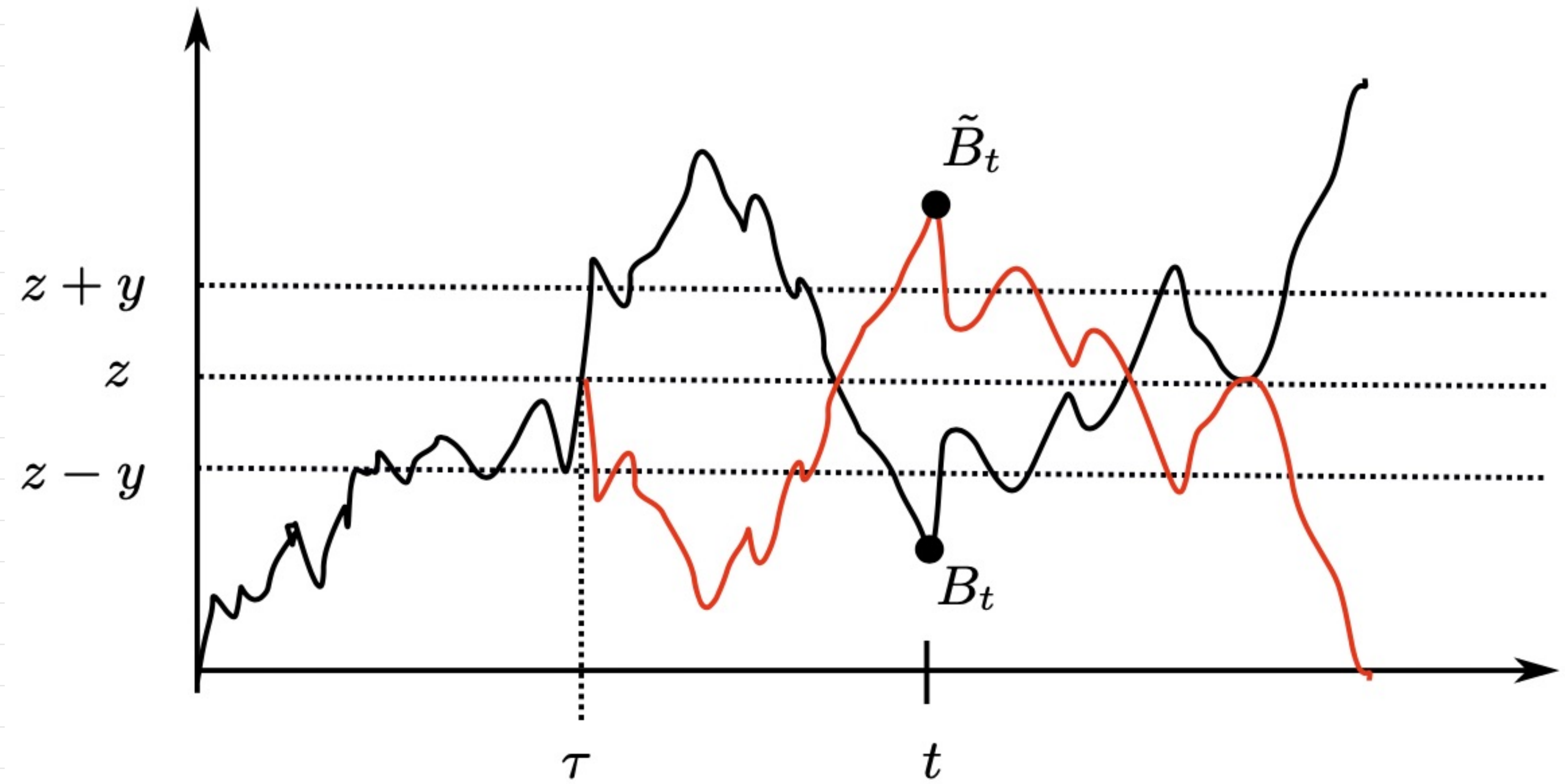
In fact, we'll compute (most of) the joint distribution of (B_t^{max}, B_t):

$$\mathbb{P}^0(B_t^{max} \geq z, B_t < z-y)$$

Note: pathwise, the processes $(B_t^{max})_{t\geq 0}$, $(|B_t|)_{t\geq 0}$ are very different.

Theorem: Let $z>0$, $y\geq 0$. $\mathbb{P}^0(B_t^{max}\geq z,\ B_t<z-y)=\mathbb{P}(B_t>z+y)$, $\therefore$ $B_t^{max}\overset{d}{=}|B_t|$.

Pf. Let $\tau=T_z$, the $B_\cdot$-Hitting time of $\{z\}$



Let $(\tilde{B}_t)_{t\geq 0}$ be the Brownian motion reflected at time τ,

$\tilde{B}_t=B_{t\wedge\tau}-(B_t-B_{t\wedge\tau})$.

Then on $\{\tau\leq t\}$, $\{B_t<z-y\}$

Note: $B_t=\tilde{B}_t$ for $t\leq\tau$;

$\therefore\ \tilde{\tau}=\inf\{t>0:\tilde{B}_t=z\}$

Thus $\mathbb{P}^0(B_t^{max} \geq z, B_t < z-y) = \mathbb{P}(B_t > z+y)$ for $z > 0$, $y \geq 0$.

In particular,

$$\mathbb{P}^0(B_t^{max} \geq z) = \mathbb{P}^0(B_t^{max} \geq z, B_t \geq z) + \mathbb{P}^0(B_t^{max} \geq z, B_t < z)$$

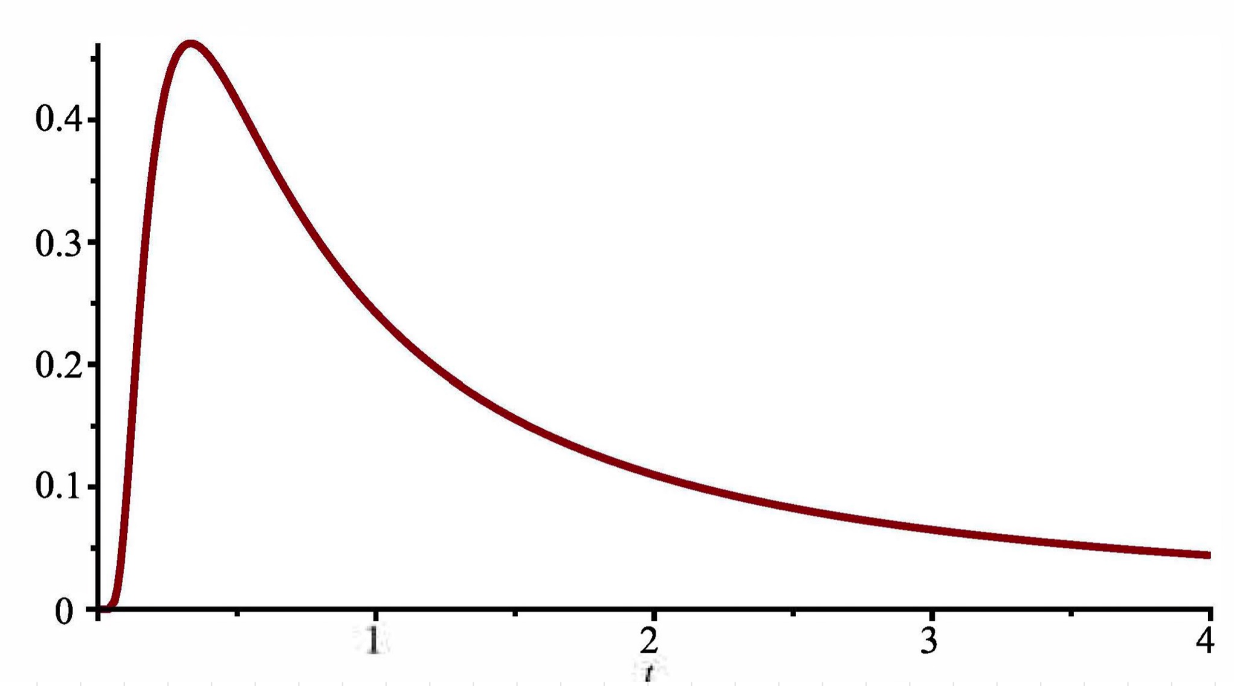
Cor: The hitting time T_z of $\{z\}$ satisfies

$$\mathbb{P}^0(T_z \leq t)$$

That is: we've explicitly calculated the probability density of T_z !

$$\rho_{T_z}(t) = \frac{z}{\sqrt{2\pi t^3}} e^{-z^2/2t} \mathbb{1}_{(0,\infty)}(t).$$

ρ_{T_1}



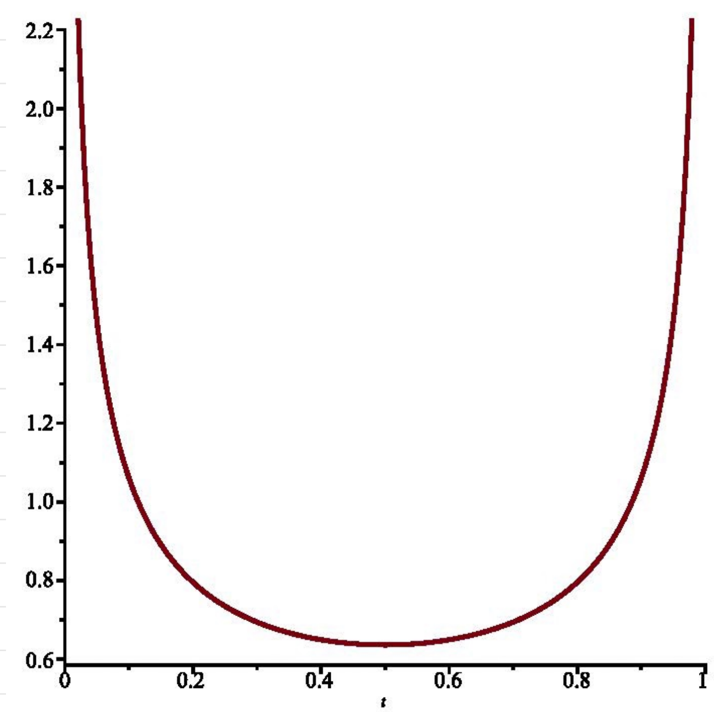
In particular, $\mathbb{P}^0(T_z < \infty) = 1 \quad \forall z$

But
$$\mathbb{E}^0[T_z] = \int_0^\infty t \cdot \rho_{T_z}(t)\,dt$$
$$= \frac{z}{\sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{t}} e^{-z^2/2t}\,dt$$

We can use this density to compute other distributions exactly.

Prop: (Arcsine Law) Let $L_1 = \sup\{0 \le t \le 1 : B_t = 0\}$.

Then $\mathbb{P}(L_1 \le t) = \frac{2}{\pi} \arcsin(\sqrt{t})$, $0 \le t \le 1$.



(Note: $B^\alpha_t := \frac{1}{\sqrt{\alpha}} B_{\alpha t}$ is also a Brownian motion, and

$L_\alpha^{B^\alpha}$

Pf. $\mathbb{P}^0(L_1 \le t) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}\, \mathbb{P}^x(T_0 > 1-t)\, dx$ [HW]

$$= 2 \cdot \frac{1}{\sqrt{2\pi t}} \int_0^\infty e^{-x^2/2t}\, dx \int_{1-t}^\infty \frac{x}{\sqrt{2\pi u^3}} e^{-x^2/2u}\, du$$

More fun facts:

$$L_1 = \sup\{0 \le t \le 1 : B_t = 0\} \overset{d}{=} \text{Arcsin Law}$$

$$\overset{d}{\parallel}$$

$$M_1 = \inf\{t > 0 : B_t = B_1^{max}\}$$

$$\overset{d}{\parallel}$$

$$\Lambda = |\{t \in [0,1] : B_t > 0\}|$$

(See [Kallenberg, Thm. 13.16].)