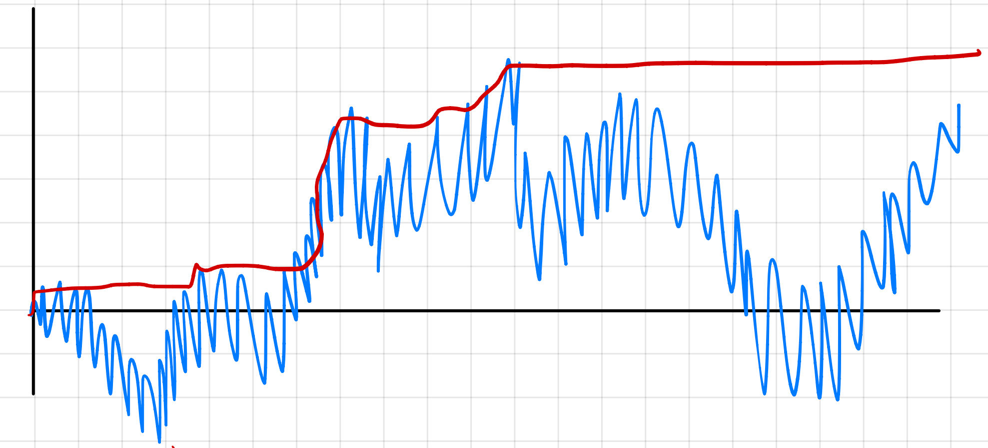
Let $(B_t)_{t\geq 0}$ be a Brownian motion on $\mathbb{R}$.



Consider the running maximum $B_t^{max} = \max_{s\leq t} B_s$.

For discrete-time submartingales, we proved several useful estimates for the values of running maxima $B_t^* = |B|_t^{max}$.
We could (given more time) prove the same results for continuous-time martingales like Brownian motion. But we can prove stronger results here.

Theorem: (Bachelier) For each $t\geq 0$, $B_t^{max} \overset{d}{=} |B_t|$.

In fact, we'll compute (most of) the joint distribution of (B_t^{max}, B_t):

$$\mathbb{P}^0(B_t^{max} \geq z,\ B_t < z-y) = \mathbb{P}^0(B_t > z+y)$$

$z \geq 0$, $y \geq 0$.

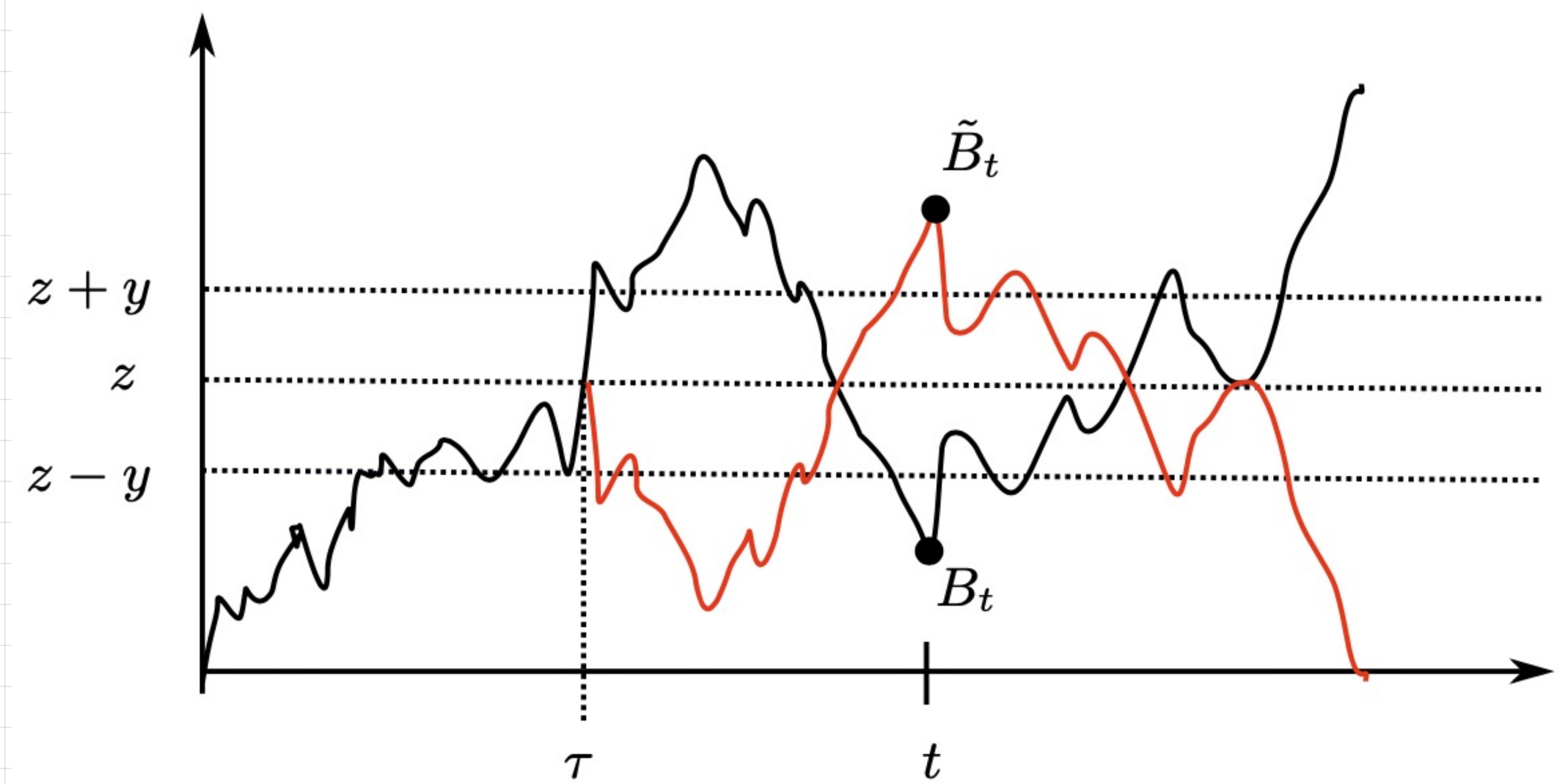
$$= \mathbb{P}\left(Z > \frac{z+y}{\sqrt{t}}\right) \qquad Z \overset{d}{=} N(0,1)$$

Note: pathwise, the processes $(B_t^{max})_{t\geq 0}$, $(|B_t|)_{t\geq 0}$ are very different.

Bessel process as rough as B_t.

Theorem: Let $z>0$, $y\geq 0$. $\mathbb{P}^0(B_t^{max}\geq z, B_t<z-y)=\mathbb{P}(B_t>z+y)$, $\therefore B_t^{max}\overset{d}{=}|B_t|$.

Pf. Let $\tau=T_z$, the $B_\cdot$-Hitting time of $\{z\}$ ← closed $\therefore$ optional time.



Let $(\tilde{B}_t)_{t\geq 0}$ be the Brownian motion reflected at time τ,

$$\tilde{B}_t=B_{t\wedge\tau}-(B_t-B_{t\wedge\tau}).$$

↑ Another Brownian motion, by Reflection Principle.

Then on $\{\tau\leq t\}$, $\{B_t<z-y\}=\{\tilde{B}_t>z+y\}\subset\{\tilde{\tau}\leq t\}$

$$\{\max_{s\leq t}B_s\geq z\}=\{B_t^{max}\geq z\}$$

$\therefore \{B_t^{max}\geq z, B_t<z-y\}=\{\tau\leq t, \tilde{B}_t>z+y\}$

$=\{\tilde{\tau}\leq t, \tilde{B}_t>z+y\}$

$=\{\tilde{B}_t>z+y\}$

Note: $B_t=\tilde{B}_t$ for $t\leq\tau$;

$\therefore \tilde{\tau}=\inf\{t>0:\tilde{B}_t=z\}$

$=\tau$.

Thus $\mathbb{P}^0(B_t^{max} \geq z, B_t < z-y) = \mathbb{P}^0(B_t > z+y)$ for $z>0, y \geq 0$.

In particular,

$$\mathbb{P}^0(B_t^{max} \geq z) = \mathbb{P}^0(B_t^{max} \geq z, B_t \geq z) + \mathbb{P}^0(B_t^{max} \geq z, B_t < z-0)$$

$$= \mathbb{P}^0(B_t \geq z) + \mathbb{P}^0(B_t > z+0)$$

$$= 2\,\mathbb{P}(B_t > z) = \mathbb{P}(B_t > z) + \mathbb{P}(B_t < -z) \quad \text{b/c } B_t \overset{d}{=} -B_t.$$

$$= \mathbb{P}(|B_t| > z)$$

$\mathbb{P}^0(B_t^* \geq z)$

$\leq \mathbb{P}^0(B_t^{max} \geq z)$

$+\mathbb{P}^0((-B_t)^{max} \geq z)$

$= 4\,\mathbb{P}(|B_t| \geq z)$

$\leq 2e^{-z^2/2t}$.

Cor: The hitting time T_z of $\{z\}$ satisfies

$$\mathbb{P}^0(T_z \leq t) = \mathbb{P}^0(B_t^{max} \geq z)$$

subs. $x = z\sqrt{\frac{t}{u}}$

$$\therefore\ 2\mathbb{P}^0(B_t > z) = 2\int_z^\infty \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}\,dx$$

$$= 2\int_t^0 \frac{1}{\sqrt{2\pi t}} e^{-z^2/2u}\left(z\sqrt{t}\cdot\left(-\tfrac{1}{2}\right)\right)u^{-3/2}\,du$$

$$= \int_0^t \frac{z}{\sqrt{2\pi u^3}} e^{-z^2/2u}\,du$$

That is: we've explicitly calculated the probability density of T_z !

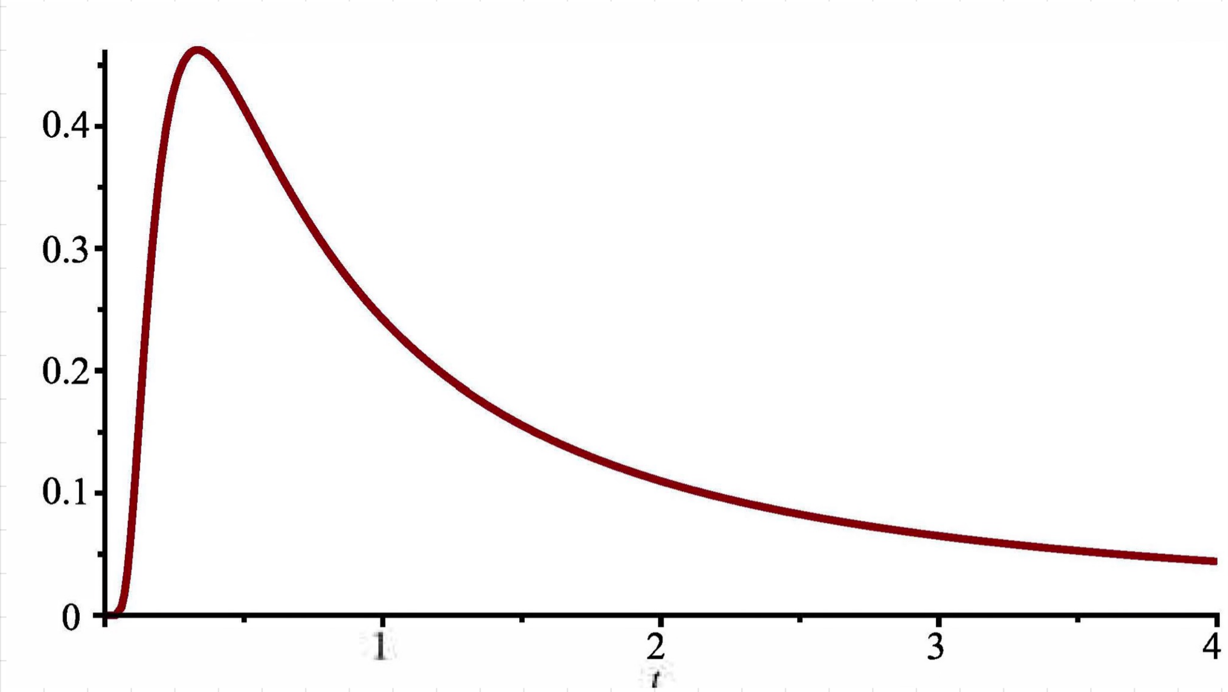
$$\rho_{T_z}(t) = \frac{z}{\sqrt{2\pi t^3}}\, e^{-z^2/2t}\, \mathbb{1}_{(0,\infty)}(t).$$

$\leftarrow$ $\alpha > 0$,

$\rho_{T_{\alpha z}}(\alpha t) = \frac{1}{\alpha}\rho_{T_z}(t)$

$\therefore\ \rho_{T_{\sqrt{\alpha}}}(s) = \frac{1}{\alpha}\rho_{T_1}(s/\alpha)$

ρ_{T_1}



In particular, $\mathbb{P}^0(T_z < \infty) = 1 \quad \forall z$ $\qquad T^B_{-z} = T^{-B}_{z} \overset{d}{=} T^B_z$

(Already knew this: $\limsup_{t\to\infty} \pm B_t = \infty$ a.s.)

But $\mathbb{E}^0[T_z] = \int_0^\infty t\cdot \rho_{T_z}(t)\,dt$

$$= \frac{z}{\sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{t}}\, \underbrace{e^{-z^2/2t}}_{\substack{\text{bounded in } [0,1] \\ \to 1 \text{ as } t\to\infty}}\, dt \sim \int_0^\infty \frac{1}{\sqrt{t}}\,dt = \infty.$$

$\therefore$ 1-dim B.M. is null-recurrent!

We can use this density to compute other distributions exactly.

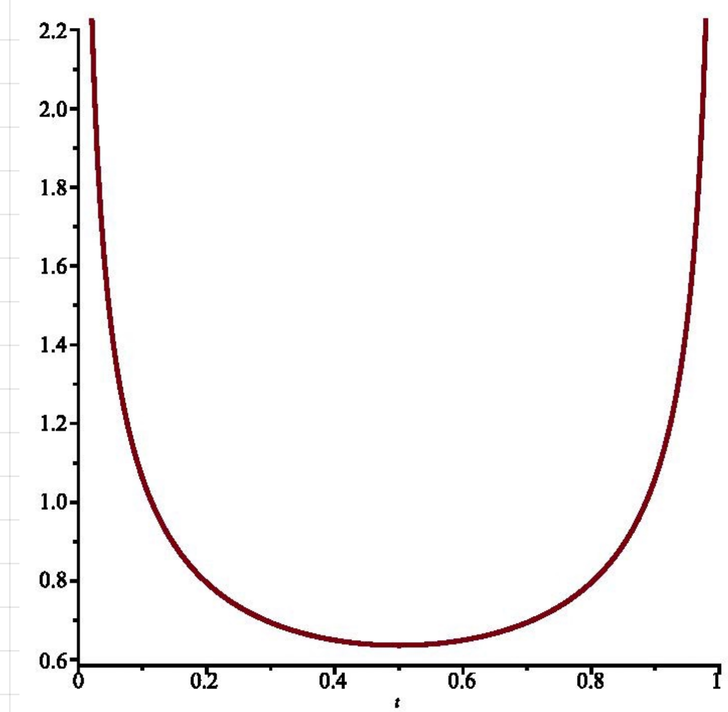
Prop. (Arcsine Law) Let $L_1 = \sup\{0 \le t \le 1 : B_t = 0\}$.

$L_1 \overset{d}{=} \sin^2 U,$

$U \overset{d}{=} \mathrm{Unif}\,[0, 2\pi]$

Then $\mathbb{P}(L_1 \le t) = \frac{2}{\pi}\arcsin(\sqrt{t}), \quad 0 \le t \le 1.$

$$\rho_{L_1}(r) = \frac{1}{\pi}\frac{1}{\sqrt{r(1-r)}}\mathbb{1}_{(0,1)}(r)$$



(Note: $B^\alpha_t := \frac{1}{\sqrt{\alpha}} B_{\alpha t}$ is also a Brownian motion, and

$$L^B_\alpha \overset{d}{=} L^{B^\alpha}_\alpha = \sup\{t \le \alpha : B^\alpha_t = 0\} = \sup\{t \le \alpha : B_{\alpha t} = 0\}$$
$$= \alpha \cdot \sup\{s \le 1 : B_s = 0\} = \alpha L^B_1)$$

Pf. $$\mathbb{P}^0(L_1 \le t) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}\, \mathbb{P}^x(T_0 > 1-t)\, dx \quad \text{[HW]}$$
$$= \mathbb{P}^0(T_x > 1-t)$$

$$= 2 \cdot \frac{1}{\sqrt{2\pi t}} \int_0^\infty e^{-x^2/2t}\, dx \int_{1-t}^\infty \frac{x}{\sqrt{2\pi u^3}} e^{-x^2/2u}\, du$$

calculus $\downarrow$

$$= \frac{1}{\pi}\int_{1-t}^\infty \frac{\sqrt{t}}{\sqrt{u}\,(t+u)}\, du = \frac{1}{\pi}\int_0^t \frac{dr}{\sqrt{r(1-r)}}.$$

sub. $r = t/(u+t)$ ///

More fun facts:

$$L_1 = \sup\{0 \le t \le 1 : B_t = 0\} \overset{d}{=} \text{Arcsin Law}$$

$$\overset{d}{\shortparallel}$$

$$M_1 = \inf\{t > 0 : B_t = B_1^{max}\}$$

$$\overset{d}{\shortparallel}$$

$$\Lambda = |\{t \in [0,1] : B_t > 0\}|$$

(See [Kallenberg, Thm. 13.16].)