We saw in [Lec 55.2] that if $(B_t)_{t\geq 0}$ is a Brownian motion on $\mathbb{R}^d$ and $T\geq 0$, then $S_t := B_{T+t} - B_T$ is a Brownian motion on $\mathbb{R}^d$, independent from $\mathcal{F}_T$

Prop: Let τ be an optional time, $\nu \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ s.t. $\mathbb{P}^\nu(\tau<\infty) > 0$, and define $S_t := B_{\tau+t} - B_\tau$ on $\{\tau<\infty\}$.
Then conditioned on $\{\tau<\infty\}$, $(S_t)_{t\geq 0}$ is a Brownian motion on $\mathbb{R}^d$, independent from $\mathcal{F}_\tau^+$.

To be precise: $\forall F \in \mathbb{B}(C([0,\infty),\mathbb{R}^d), \mathcal{C}([0,\infty),\mathbb{R}^d))$

$$E^\nu[F(S) \mid \tau<\infty] = E^0[F(B)]$$

and $\forall A \in \mathcal{F}_\tau^+$,

$$E^\nu[F(S)\mathbb{1}_A \mid \tau<\infty] = E^\nu[F(S) \mid \tau<\infty]\,\mathbb{P}^\nu(A \mid \tau<\infty)$$

Pf. Abusing notation slightly, denote $X_{\tau+\cdot} = X_\cdot \circ \theta_\tau$ for any process X.

Then $S_t = B_{t+\tau} - B_\tau$

Hence $\mathbb{E}^\nu[F(S)\mathbb{1}_{\{\tau<\infty\}} \mid \mathcal{F}_\tau^+]$

$$= \mathbb{E}^\nu[F((B_\cdot - B_0)\circ\theta_\tau) \mid \mathcal{F}_\tau^+]\mathbb{1}_{\{\tau<\infty\}}$$

Thus, if $A \in \mathcal{F}_\tau^+$,

$$\mathbb{E}^\nu[F(S)\mathbb{1}_A \mid \tau<\infty] = \frac{1}{\mathbb{P}^\nu(\tau<\infty)}\mathbb{E}^\nu[\mathbb{1}_{\{\tau<\infty\}}F(S)\mathbb{1}_A]$$

$\therefore \quad E^{\nu}[F(S)\mathbb{1}_A \mid \tau<\infty] = E^{0}[F(B)]\,E^{\nu}[\mathbb{1}_A \mid \tau<\infty] \quad \forall A \in \mathcal{F}_\tau^+ .$

Take $A=\Omega$; shows

Note: in [Lec. 55.2] we showed $B_{T+\cdot} - B_T$ is a Brownian motion indep. from $\mathcal{F}_T$
when $T \geq 0$ is constant. Now we've proved the stronger claim
that it is independent from $\mathcal{F}_T^+$

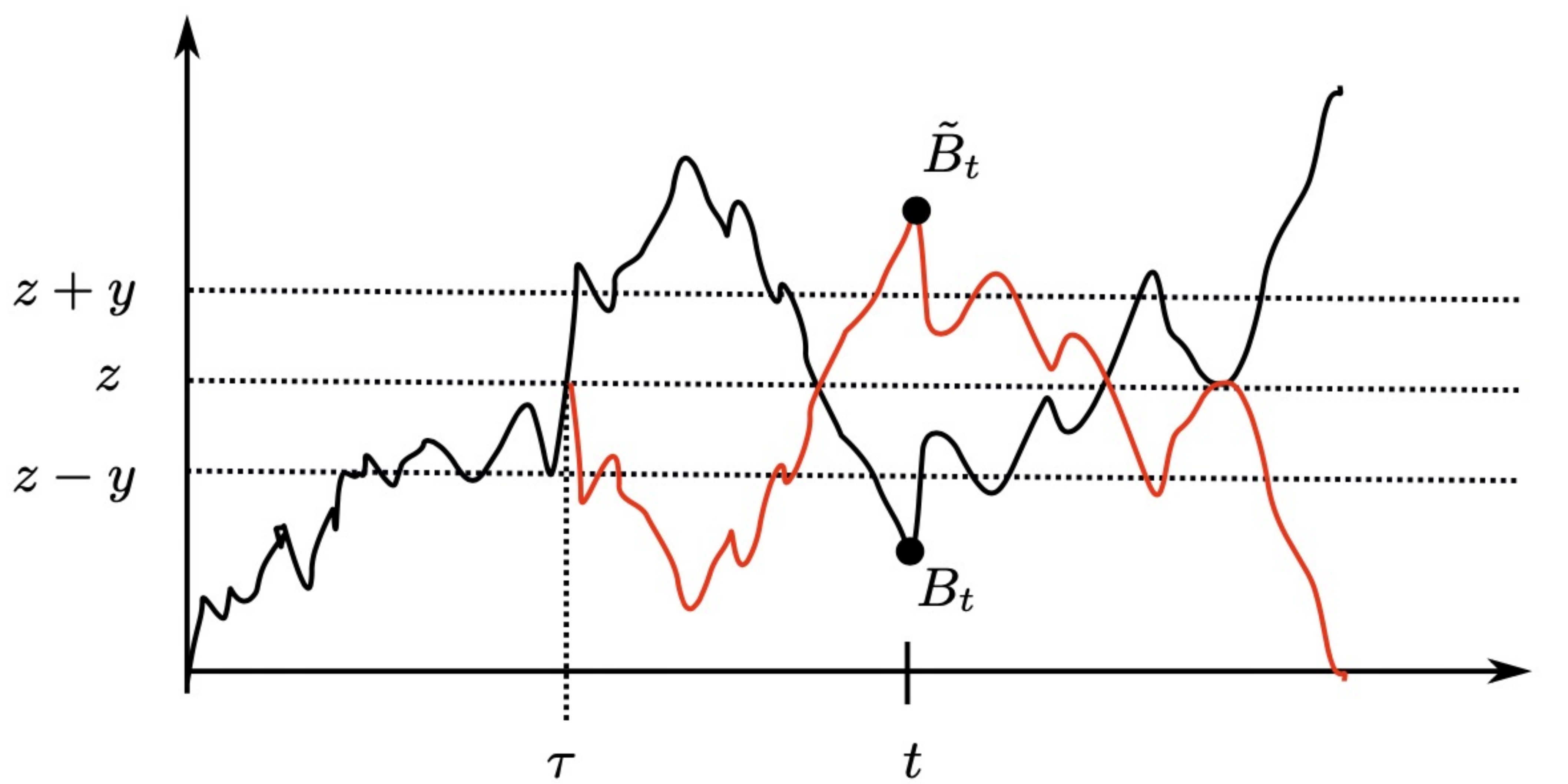
Also: if τ is an optional time, it is $\mathcal{F}_\tau^+$-measurable.
Hence: $S_t = B_{t+\tau} - B_\tau$ is indep. from τ.
(conditioned on $\tau<\infty$).

Theorem: (Reflection Principle)

Let B be a Brownian motion, and let τ be an optional time adapted to its natural filtration. Then

$$\tilde{B}_t := B_{t\wedge\tau} - (B_t - B_{t\wedge\tau}), \quad t \geq 0$$

is a Brownian motion.



(Reflection Principle) $\tilde{B}_t := B_{t\wedge\tau} - (B_t - B_{t\wedge\tau})$ is a Brownian motion.

Pf. It suffices to show $\tilde{B}|_{[0,T]}$ is a Brownian motion for each $T>0$.

$\therefore$ Replacing τ with $\tau\wedge T$ if needed, wlog assume $\tau<\infty$.

We $\therefore$ know that $S_t = B_{t+\tau} - B_\tau$ is a Brownian motion, indep. from $\mathcal{F}_\tau^+$.

$$\left.\begin{array}{l} \bullet\ \tau \text{ is } \mathcal{F}_\tau^+\text{-measurable} \\ \bullet\ B_t^\tau = B_{t\wedge\tau} \text{ is } \mathcal{F}_\tau^+\text{-measurable} \end{array}\right\} \text{[Lec. 56.3]}$$

Thus S is independent from (τ, B^τ).

Now note that $S_{(t-\tau)_+}$